CHBE320 LECTURE X
STABILITY OF CLOSED-LOOP CONTOL SYSTEMS

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Road Map of the Lecture X

- Stability of closed-loop control system
  - Definition
    - General stability
    - BIBO stanility
  - Stability criteria
    - Routh-Hurwitz stability criterion
    - Direct substitution method
    - Root locus method
    - Bode stability criterion
    - Gain margin and Phase margin
    - Nyquist stability criterion
  - Robustness

DEFINITION OF STABILITY

- **BIBO Stability**
  - **"An unconstrained linear system is said to be stable if the output response is bounded for all bounded inputs. Otherwise it is said to be unstable."**
- **General Stability**
  - **A linear system is stable if and only if all roots (poles) of the denominator in the transfer function are negative or have negative real parts (OLHP). Otherwise, the system is unstable.**

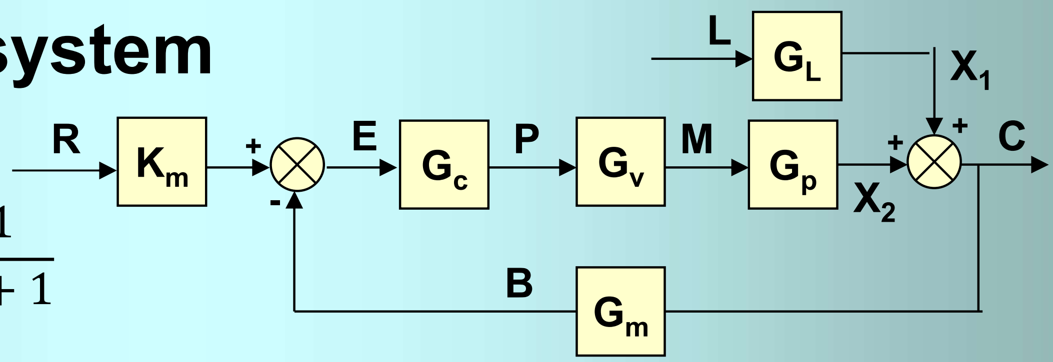
$\Rightarrow$ **What is the difference between the two definitions?**

- **Open-loop stable/unstable**
- **Closed-loop stable/unstable**
- **Characteristic equation:** $1 + G_{OL}(s) = 0$
- **Nonlinear system stability: Lyapunov and Popov stability**

- **Supplements for stability**
  - For input-output model,
    - Asymptotic stability (AS): For a system with zero equilibrium point, if $u(t)=0$ for all time t implies $y(t)$ goes to zero with time.
      - Same as "General stability": all poles have to be in OLHP.
    - Marginally stability (MS): For a system with zero equilibrium point, if $u(t)=0$ for all time t implies $y(t)$ is bounded for all time.
      - Same as BIBO stability: all poles have to be in OLHP or on the imaginary axis with any poles occurring on the imaginary axis non-repeated.
      - If the imaginary pole is repeated the mode is $t\sin(wt)$ and it is unstable.
  - For state-space model,
    - Even though there are unstable poles and if they are cancelled by the zeros exactly (pole-zero cancellation), the system is BIBO stable.
    - Internally AS: if $u(t)=0$ for all time, it implies that $x(t)$ goes to zero with time for all initial conditions $x(0)$.
      - Cancelled poles have to be in OLHP.

EXAMPLES

- **Feedback control system**

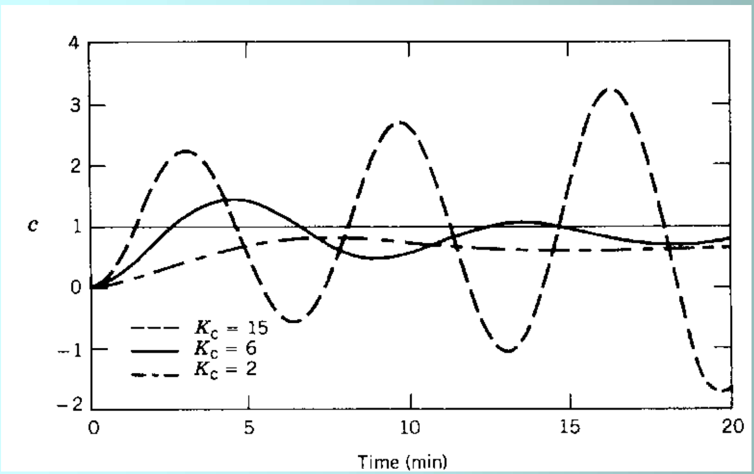


$$G_c(s) = K_c$$

$$G_v(s) = \frac{1}{2s+1} \qquad G_m(s) = \frac{1}{s+1}$$

$$G_p(s) = G_L(s) = \frac{1}{5s+1}$$

$$\frac{C(s)}{R(s)} = \frac{K_m G_c G_v G_p}{1 + G_c G_v G_p G_m} = \frac{K_c(s+1)}{10s^3 + 17s^2 + 8s + 1 + K_c}$$



- **Using root-finding techniques, the poles can be calculated.**
- **As K_c increases, the step response gets more oscillatory.**
- **If K_c >12.6, the step response is unstable.**

- **Simple Example 1**

$G_c(s) = K_c, \quad G_v(s) = K_v, \quad G_m(s) = 1, \quad G_p(s) = K_p/(\tau_p s + 1)$

Characteristic equation: $1 + G_{OL}(s) = 1 + K_c K_v K_p/(\tau_p s + 1) = 0$

$\tau_p s + (1 + K_c K_v K_p) = 0 \quad \Rightarrow \quad s = -(1 + K_c K_v K_p)/\tau_p$

$\therefore \; K_c K_v K_p > -1$ for stability

  – **When K_p>0 and K_v>0, the controller should be reverse acting (K_c>0) for stability.**

- **Simple example 2**

$G_c(s) = K_c, \quad G_v(s) = 1/(2s + 1), \quad G_m(s) = 1, \quad G_p(s) = 1/(5s + 1)$

Characteristic equation: $1 + K_c/[(2s + 1)(5s + 1)] = 0$

$10s^2 + 7s + 1 + K_c = 0 \quad \Rightarrow \quad s = \left[-7 \pm \sqrt{49 - 40(1 + K_c)}\right]/20$

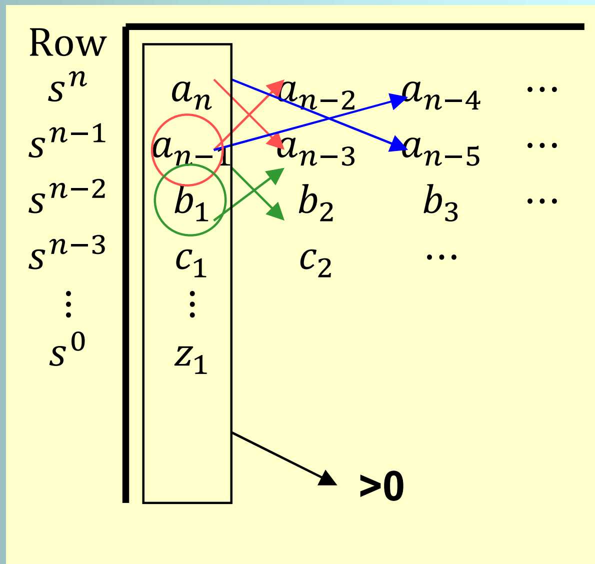
$\therefore \; K_c > -1$ for stability

ROUTH-HURWITZ STABILITY CRITERION

- **From the characteristic equation of the form:**

$$a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 = 0 \quad (a_n > 0)$$

- **Construct the Routh array**



$$b_1 = (a_{n-1}a_{n-2} - a_n a_{n-3})/a_{n-1}$$
$$b_2 = (a_{n-1}a_{n-4} - a_n a_{n-5})/a_{n-1}$$
$$\vdots$$
$$c_1 = (b_1 a_{n-3} - a_{n-1}b_2)/b_1$$
$$c_2 = (b_1 a_{n-5} - a_{n-1}b_3)/b_1$$
$$\vdots$$

$$b_1 = -\begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix} / a_{n-1}$$

$$b_2 = -\begin{vmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{vmatrix} / a_{n-1}$$

$$c_1 = -\begin{vmatrix} a_{n-1} & a_{n-3} \\ b_1 & b_2 \end{vmatrix} / b_1$$

$$c_2 = -\begin{vmatrix} a_{n-1} & a_{n-5} \\ b_1 & b_3 \end{vmatrix} / b_1$$

- **A necessary condition for stability: all a_i's are positive**
- **"A necessary and sufficient condition for all roots of the characteristic equation to have negative real parts is that all of the elements in the left column of the Routh array are positive."**

Example for Routh test

- Characteristic equation

$$10s^3 + 17s^2 + 8s + 1 + K_c = 0$$

- Necessary condition

$$1 + K_c > 0 \Rightarrow K_c > -1$$

  - If any coefficient is not positive, stop and conclude the system is unstable. (at least one RHP pole, possibly more)

- Routh array

s^3	10	8
s^2	17	$1+K_c$
s^1	b_1	b_2
s^0	c_1	

$$b_1 = \frac{17(8) - 10(1+K_c)}{17} = 7.41 - 0.588K_c$$

$$b_2 = \frac{17(0) - 10(0)}{17} = 0$$

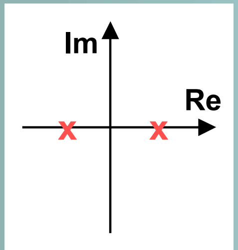
$$c_1 = \frac{b_1(1+K_c) - 17(0)}{b_1} = 1 + K_c$$

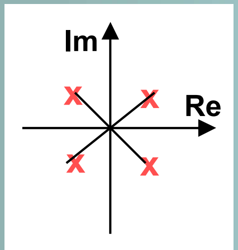
- Stable region

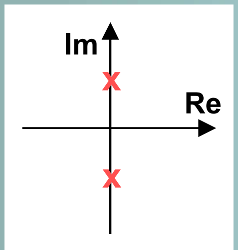
$$b_1 = 7.41 - 0.588K_c > 0 \text{ and } K_c > -1 \quad \Rightarrow \quad -1 < K_c < 12.6$$

- **Supplements for Routh test**
  - It is valid only when the characteristic equation is a polynomial of s. (Time delay cannot be handled directly.)
  - If the characteristic equation contains time delay, use Pade approximation to make it as a polynomial of s.
  - Routh test can be used to test if the real part of all roots of characteristic equation are less than -c.
    - Original characteristic equation
      $$a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 = 0 \quad (a_n > 0)$$
    - Modify characteristic equation and apply Routh criterion
      $$a_n (s+c)^n + a_{n-1}(s+c)^{n-1} + \cdots + a_1(s+c) + a_0$$
      $$= a'_n s^n + a'_{n-1} s^{n-1} + \cdots + a'_1 s + a'_0 = 0$$
  - The number of sign change in the 1st column of the Routh array indicates the number of poles in RHP.
  - If the two rows are proportional or any element of 1st column is zero, Routh array cannot be proceeded.

- **Remedy for special cases of Routh array**
  - **Only the pivot element is zero and others are not all zero**
    - **Replace zero with positive small number (*e*), and proceed.**
    - **If there is no sign change in the 1st column, it indicates there is a pair of pure imaginary roots (marginally stable). If not, the sign change indicates the no. of RHP poles.**
  - **Entire row becomes zero (two rows are proportional)**
    - **It implies the characteristic polynomial is divided exactly by the polynomial one row above (always even-ordered polynomial).**
    - **Replace the row with the coefficients of the derivative (auxiliary polynomial) of the polynomial one row above and proceed.**
    - **This situation indicates at least either a pair of real roots symmetric about the origin (one unstable), and/or two complex pairs symmetric about the origin (one unstable pair).**
    - **If there is no sign change after the auxiliary polynomial, it indicated that the polynomial prior to the auxiliary polynomial has all pure imaginary roots.**







DIRECT SUBSTITUTION METHOD

- **Find the value of variable that locates the closed-loop poles at the imaginary axis (stability limit).**
- **Example**
  - **Characteristic equation:** $1 + G_c G_p = 10s^3 + 17s^2 + 8s + 1 + K_c = 0$
  - **On the imaginary axis *s* becomes** $j\omega$**.**

$$-10j\omega^3 - 17\omega^2 + 8j\omega + 1 + K_{cm} = (1 + K_{cm} - 17\omega^2) + j\omega(8 - 10\omega^2) = 0$$

$$\therefore (1 + K_{cm} - 17\omega^2) = 0 \quad \text{and} \quad \omega(8 - 10\omega^2) = 0$$

$$\therefore \omega = 0 \text{ or } \omega^2 = 0.8 \Rightarrow K_{cm} = -1 \text{ or } K_{cm} = 12.6$$

  - **Try a test point such as K_c=0**

$$10s^3 + 17s^2 + 8s + 1 = (s+1)(2s+1)(5s+1) = 0 \ \text{(All stable)}$$

$$\therefore \text{Stable range is } -1 < K_c < 12.6$$

ROOT LOCUS DIAGRAMS

- **Diagram shows the location of closed-loop poles (roots of characteristic equation) depending on the parameter value. (Single parametric study)**
- **Find the roots as a function of parameter**
- **Each loci starts at open-loop poles and approaches to zeros or $\pm\infty$.**

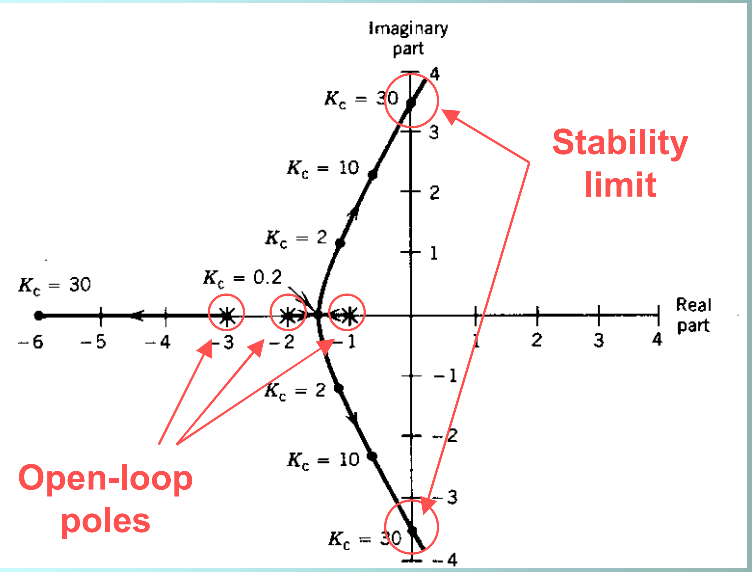
For $G(s) = N(s)/D(s)$

$$\lim_{K_c \to 0} (D(s) + K_c N(s)) = D(s) = 0 \text{ (poles)}$$

$$\lim_{K_c \to \infty} (D(s) + K_c N(s)) = N(s) = 0 \text{ (zeros)}$$

Ex) $(s+1)(s+2)(s+3) + 2K_c = 0$

$$\Rightarrow K_c < 30$$



BODE STABILITY CRITERION

- **Bode stability criterion**

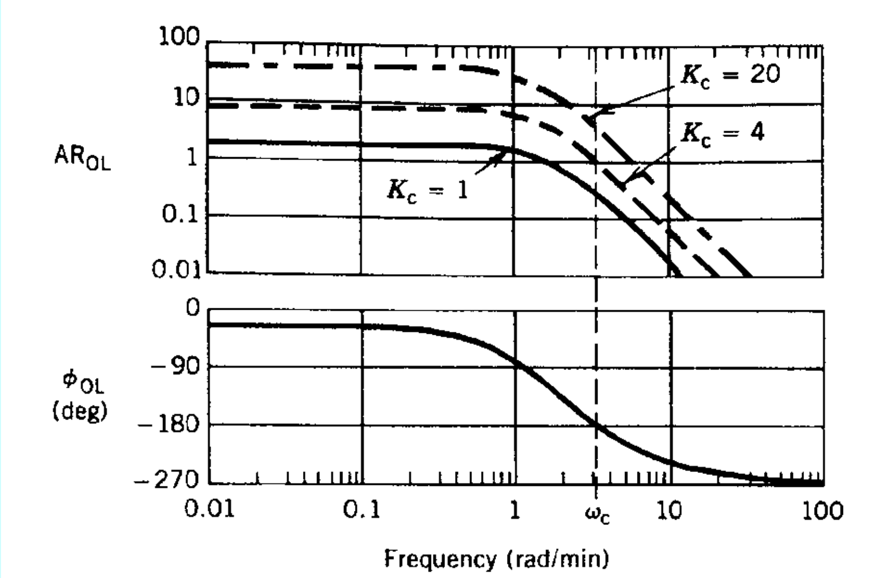
  "A closed-loop system is unstable if the frequency response of the open-loop transfer function $G_{OL}=G_c G_v G_p G_m$ has an amplitude ratio greater than one at the critical frequency. Otherwise, the closed-loop system is stable."

  – Applicable to open-loop stable systems with only one critical frequency

  – Example:

$$G_{OL} = \frac{2K_c}{(0.5s+1)^3}$$

K_c	AR_{OL}	Classification
1	0.25	stable
4	1	Marginally stable
20	5	unstable



GAIN MARGIN (GM) AND PHASE MARGIN (PM)

- Margin: How close is a system to stability limit?
- Gain Margin (GM)

  $$GM = 1/AR(\omega_c)$$

  – For stability, GM>1

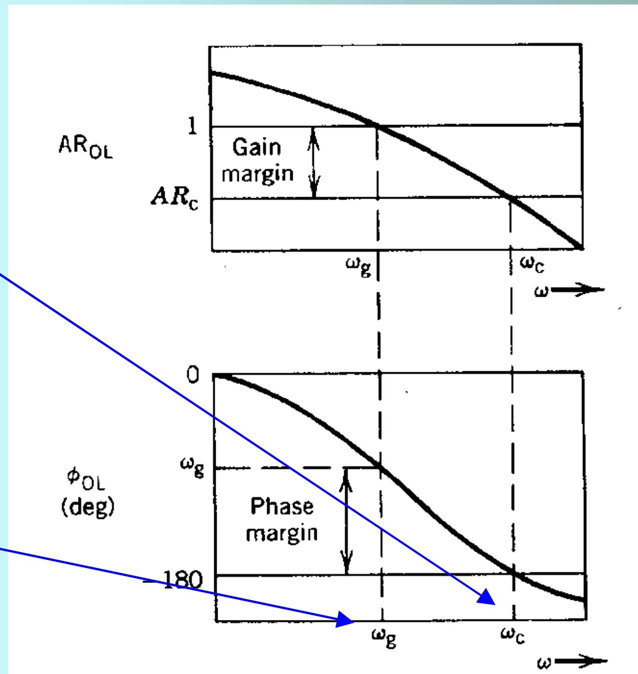
- Phase Margin (PM)

  $$PM = \varphi(\omega_g) + 180°$$

  – For stability, PM>0

Critical or Phase crossover freq.

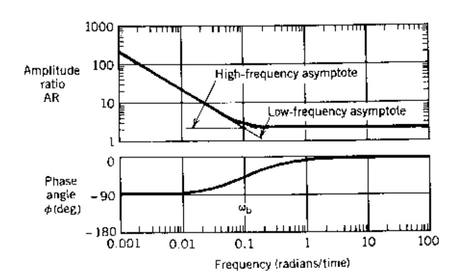
Gain crossover freq.



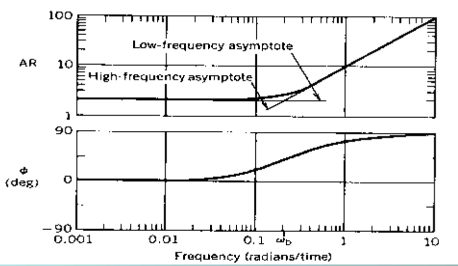
- Rule of thumb
  - Well-tuned system: GM=1.7 - 2.0,   PM=30° - 45°
  - Large GM and PM: sluggish
  - Small GM and PM: oscillatory
  - If the uncertainty on process is small, tighter tuning is possible.

EFFECT OF PID CONTROLLERS ON FREQUENCY RESPONSE

- **P**
  - As K_c increases, AR_{OL} increases (faster but destabilizing)
  - No change in phase angle
- **PI**



  - Increase AR_{OL} more at low freq.
  - As τ_I decreases, AR_{OL} increases (destabilizing)
  - More phase lag for lower freq. (moves critical freq. toward lower freq. => usually destabilizing)
- **PD**



  - Increase AR_{OL} more at high freq.
  - As τ_D increases, AR_{OL} increases at high freq. (faster)
  - More phase lead for high freq. (moves critical freq. toward higher freq. => usually stabilizing)

NYQUIST STABILITY CRITERION

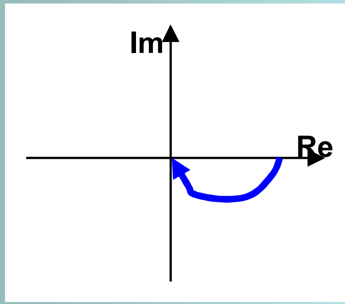
- **Nyquist stability criterion**

  "If N is the number of times that the Nyquist plot encircles the point (-1,0) in the complex plane in the clockwise direction, and P is the number of open-loop poles of $G_{OL}(s)$ that lies in RHP, then $Z=N+P$ is the number of unstable roots of the closed-loop characteristic equation."

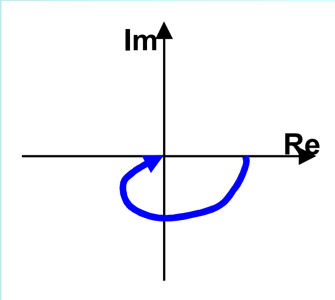
  - **Applicable to even unstable systems and the systems with multiple critical frequencies**
  - **The point (-1,0) corresponds to AR=1 and PA=-180°.**
  - **Negative N indicates the encirclement of (-1,0) in counterclockwise direction.**

- **Some examples**

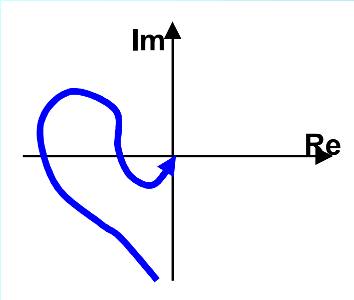
1st order lag



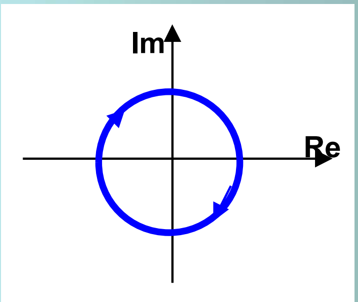
2nd order lag



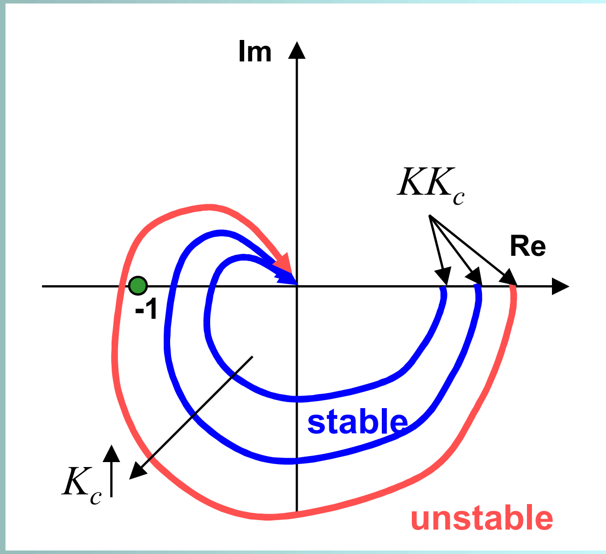
$$G(s) = \frac{K_c(2s+1)(s+1)}{s(20s+1)(10s+1)(0.5s+1)}$$



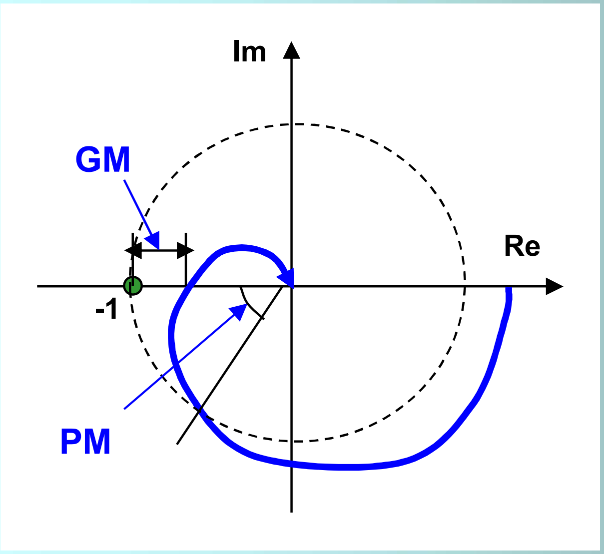
Pure time delay



Stable 3rd order lag + P control



GM and PM

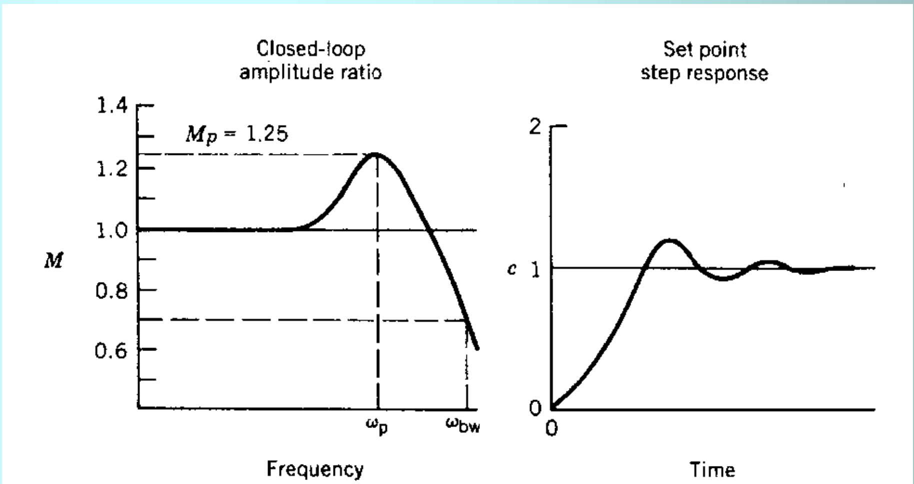


CLOSED-LOOP FREQUENCY RESPONSE

- **Closed-loop amplitude ratio and phase angle**

$$M = \left|\frac{Y(j\omega)}{R(j\omega)}\right|$$

$$\psi = \measuredangle \frac{Y(j\omega)}{R(j\omega)}$$



For set point change,

- M **should be unity as** $\omega \rightarrow 0$ **. (No offset)**
- M **should maintain at unity up to high frequency as possible. (rapid approach to a new set point)**
- **A resonant peak (**M_p**) in** M **is desirable but not greater than 1.25. (large** ω_p **implies faster response to a new set point)**
- **Large bandwidth (**ω_{bw}**) indicates a relatively fast response with a short rise time.**

ROBUSTNESS

- **Definition**

  "Despite the small change in the process or some inaccuracies in the process model, if the control system is insensitive to the uncertainties in the system and functions properly."

  - **The robust control system should be, despite the certain size of uncertainty of the model,**
    - **Stable**
    - **Maintaining reasonable performance**
  - **Uncertainty (confidence level of the model):**
    - **Process gain, Time constants, Model order, etc.**
    - **Input, output**
  - **If uncertainty is high, the performance specification cannot be too tight: might cause even instability**