CHBE320 LECTURE V
LAPLACE TRANSFORM AND TRANSFER FUNCTION

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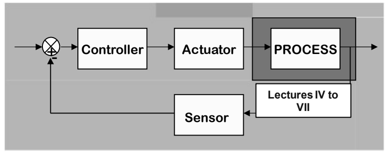
Road Map of the Lecture V

- **Laplace Transform and Transfer functions**
  - **Definition of Laplace transform**
  - **Properties of Laplace transform**
  - **Inverse Laplace transform**
  - **Definition of transfer function**
  - **How to get the transfer functions**
  - **Properties of transfer function**



SOLUTION OF LINEAR ODE

- **1st-order linear ODE**
  - **Integrating factor:** For $\frac{dx}{dt}+a(t)x=f(t)$, I.F. $=\exp(\int a(t)dt)$

    $[xe^{\int a(t)dt}]'=f(t)e^{\int a(t)dt} \longrightarrow x(t)=[\int f(t)e^{\int a(t)dt}dt+C]e^{-\int a(t)dt}$
- **High-order linear ODE with constant coeffs.**
  - **Modes: roots of characteristic equation**

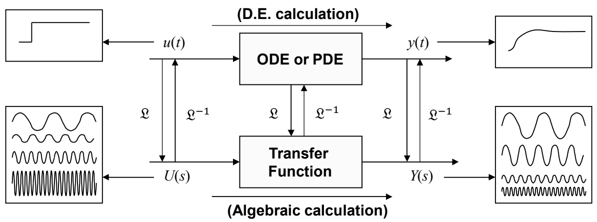
    For $a_2x''+a_1x'+a_0x=f(t)$,

    $a_2p^2+a_1p+a_0=a_2(p-p_1)(p-p_2)=0$
  - **Depending on the roots, modes are**
    - **Distinct roots:** (e^{-p_1t}, e^{-p_2t})
    - **Double roots:** (e^{-p_1t}, te^{-p_1t})
    - **Imaginary roots:** $(e^{-\alpha t}\cos\beta t, e^{-\alpha t}\sin\beta t)$

    $\longrightarrow$ **Solution is a linear combination of modes and the coefficients are decided by the initial conditions.**
- **Many other techniques for different cases**

LAPLACE TRANSFORM FOR LINEAR ODE AND PDE

- **Laplace Transform**
  - **Not in time domain, rather in frequency domain**
  - **Derivatives and integral become some operators.**
  - **ODE is converted into algebraic equation**
  - **PDE is converted into ODE in spatial coordinate**
  - **Need inverse transform to recover time-domain solution**



DEFINITION OF LAPLACE TRANSFORM

- **Definition**

$$\boxed{F(s) = \mathfrak{L}\{f(t)\} \triangleq \int_0^{\infty} f(t)e^{-st}dt}$$

  – ***F(s)* is called *Laplace transform* of *f(t)*.**
  – ***f(t)* must be piecewise continuous.**
  – ***F(s)* contains no information on *f(t)* for $t < 0$.**
  – **The past information on *f(t)* (for $t < 0$) is irrelevant.**
  – **The *s* is a complex variable called "*Laplace transform variable*"**

- **Inverse Laplace transform**

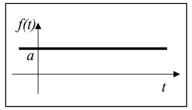
$$\boxed{f(t) = \mathfrak{L}^{-1}\{F(s)\}}$$

  – $\mathfrak{L}$ **and** $\mathfrak{L}^{-1}$ **are linear.** $\mathfrak{L}\{af_1(t) + bf_2(t)\} = aF_1(s) + bF_2(s)$

LAPLACE TRANSFORM OF FUNCTIONS

- **Constant function,** a

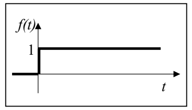
$$\mathfrak{L}\{a\} = \int_0^{\infty} ae^{-s}\,dt = -\frac{a}{s}e^{-st}\Big|_0^{\infty} = 0 - \left(-\frac{a}{s}\right) = \frac{a}{s}$$



- **Step function,** S(t)

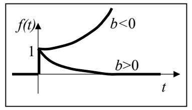
$$f(t) = \mathrm{S}(t) = \begin{cases} 1 & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

$$\mathfrak{L}\{\mathrm{S}(t)\} = \int_0^{\infty} e^{-s}\,dt = -\frac{1}{s}e^{-st}\Big|_0^{\infty} = 0 - \left(-\frac{1}{s}\right) = \frac{1}{s}$$



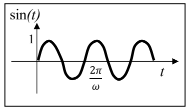
- **Exponential function,** e^{-bt}

$$\mathfrak{L}\{e^{-bt}\} = \int_0^{\infty} e^{-b}\,e^{-s}\,dt = \frac{-1}{s+b}e^{-(b+s)t}\Big|_0^{\infty} = \frac{1}{s+b}$$



- **Trigonometric functions**
  – **Euler's Identity:** $e^{j\omega t} \triangleq \cos\omega t + j\sin\omega t$

$$\cos\omega t = \frac{1}{2}\left(e^{j\omega t} + e^{-j\omega t}\right) \qquad \sin\omega t = \frac{1}{2j}\left(e^{j\omega t} - e^{-j\omega}\right)$$

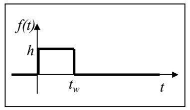


$$\mathfrak{L}\{\sin\omega t\} = \mathfrak{L}\left\{\frac{1}{2j}e^{j\omega t}\right\} - \mathfrak{L}\left\{\frac{1}{2j}e^{-j\omega t}\right\} = \frac{1}{2j}\left(\frac{1}{s-j\omega} - \frac{1}{s+j\omega}\right) = \frac{\omega}{s^2+\omega^2}$$

$$\mathfrak{L}\{\cos\omega t\} = \mathfrak{L}\left\{\frac{1}{2}e^{j\omega t}\right\} + \mathfrak{L}\left\{\frac{1}{2}e^{-j\omega}\right\} = \frac{1}{2}\left(\frac{1}{s-j\omega} + \frac{1}{s+j\omega}\right) = \frac{s}{s^2+\omega^2}$$

- **Rectangular pulse,** P(t)

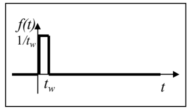
$$f(t) = \mathrm{P}(t) = \begin{cases} 0 & \text{for } t > t_w \\ h & \text{for } t_w \geq t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$



$$\mathfrak{L}\{\mathrm{P}(t)\} = \int_0^{t_w} he^{-st}dt = -\frac{h}{s}e^{-st}\Big|_0^{t_w} = \frac{h}{s}\left(1 - e^{-t_w s}\right)$$

- **Impulse function,** $\delta(t)$

$$f(t) = \delta(t) = \lim_{t_w \to 0} \begin{cases} 0 & \text{for } t > t_w \\ 1/t_w & \text{for } t_w \geq t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$



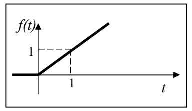
$$\mathfrak{L}\{\delta(t)\} = \lim_{t_w \to 0}\int_0^{t_w} \frac{1}{t_w}e^{-st}dt = \lim_{t_w \to 0}\frac{1}{t_w s}\left(1 - e^{-t_w s}\right) = 1$$

$$\left(\text{L'Hospital's rule: } \lim_{t \to 0}\frac{f(t)}{g(t)} = \lim_{t \to 0}\frac{f'(t)}{g'(t)}\right)$$

- **Ramp function,** t

$$\mathfrak{L}\{t\} = \int_0^{\infty} te^{-s}\,dt$$
$$= \frac{t}{-s}e^{-st}\Big|_0^{\infty} - \int_0^{\infty}\frac{e^{-s}}{-s}dt = \frac{1}{s}\int_0^{\infty}e^{-st}dt = \frac{1}{s^2}$$

$$\left(\text{Integration by part: } \int_0^{\infty} f'\cdot g\,dt = f\cdot g\Big|_0^{\infty} - \int_0^{\infty} f\cdot g'\,dt\right)$$



- **Refer the Table 3.1 (Seborg et al.) for other functions**

Table 3.1 Laplace Transforms for Various Time-Domain Functions[a]

	$f(t)$	$F(s)$
1.	$\delta(t)$ (unit impulse)	1
2.	$S(t)$ (unit step)	$\frac{1}{s}$
3.	t (ramp)	$\frac{1}{s^2}$
4.	t^{n-1}	$\frac{(n-1)!}{s^n}$
5.	e^{-bt}	$\frac{1}{s+b}$
6.	$\frac{1}{\tau}e^{-t/\tau}$	$\frac{1}{\tau s+1}$
7.	$\frac{t^{n-1}e^{-bt}}{(n-1)!}$ $(n>0)$	$\frac{1}{(s+b)^n}$
8.	$\frac{1}{\tau^n(n-1)!}t^{n-1}e^{-t/\tau}$	$\frac{1}{(\tau s+1)^n}$
9.	$\frac{1}{b_1-b_2}(e^{-b_2t}-e^{-b_1t})$	$\frac{1}{(s+b_1)(s+b_2)}$
10.	$\frac{1}{\tau_1-\tau_2}(e^{-t/\tau_1}-e^{-t/\tau_2})$	$\frac{1}{(\tau_1 s+1)(\tau_2 s+1)}$
11.	$\frac{b_3-b_1}{b_2-b_1}e^{-b_1t}+\frac{b_3-b_2}{b_1-b_2}e^{-b_2t}$	$\frac{s+b_3}{(s+b_1)(s+b_2)}$
12.	$\frac{1}{\tau_1}\frac{\tau_1-\tau_3}{\tau_1-\tau_2}e^{-t/\tau_1}+\frac{1}{\tau_2}\frac{\tau_2-\tau_3}{\tau_2-\tau_1}e^{-t/\tau_2}$	$\frac{\tau_3 s+1}{(\tau_1 s+1)(\tau_2 s+1)}$
13.	$1-e^{-t/\tau}$	$\frac{1}{s(\tau s+1)}$
14.	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$

Table 3.1 (*Continued*)

	$f(t)$	$F(s)$
15.	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
16.	$\sin(\omega t+\phi)$	$\frac{\omega\cos\phi+s\sin\phi}{s^2+\omega^2}$
17.	$e^{-bt}\sin\omega t$ (b, ω real)	$\frac{\omega}{(s+b)^2+\omega^2}$
18.	$e^{-bt}\cos\omega t$ (b, ω real)	$\frac{s+b}{(s+b)^2+\omega^2}$
19.	$\frac{1}{\tau\sqrt{1-\zeta^2}}e^{-\zeta t/\tau}\sin(\sqrt{1-\zeta^2}\,t/\tau)$ $(0\le\lvert\zeta\rvert<1)$	$\frac{1}{\tau^2s^2+2\zeta\tau s+1}$
20.	$1+\frac{1}{\tau_2-\tau_1}(\tau_1e^{-t/\tau_1}-\tau_2e^{-t/\tau_2})$ $(\tau_1\neq\tau_2)$	$\frac{1}{s(\tau_1 s+1)(\tau_2 s+1)}$
21.	$1-\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta t/\tau}\sin[\sqrt{1-\zeta^2}\,t/\tau+\psi]$ $\psi=\tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$ $(0\le\lvert\zeta\rvert<1)$	$\frac{1}{s(\tau^2s^2+2\zeta\tau s+1)}$
22.	$1-e^{-\zeta t/\tau}[\cos(\sqrt{1-\zeta^2}\,t/\tau)+\frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\sqrt{1-\zeta^2}\,t/\tau)]$ $(0\le\lvert\zeta\rvert<1)$	$\frac{1}{s(\tau^2s^2+2\zeta\tau s+1)}$
23.	$1+\frac{\tau_3-\tau_1}{\tau_1-\tau_2}e^{-t/\tau_1}+\frac{\tau_3-\tau_2}{\tau_2-\tau_1}e^{-t/\tau_2}$ $(\tau_1\neq\tau_2)$	$\frac{\tau_3 s+1}{s(\tau_1 s+1)(\tau_2 s+1)}$
24.	$\frac{df}{dt}$	$sF(s)-f(0)$
25.	$\frac{d^nf}{dt^n}$	$s^nF(s)-s^{n-1}f(0)-s^{n-2}f^{(1)}(0)-\cdots-sf^{(n-2)}(0)-f^{(n-1)}(0)$

[a] Note that $f(t)$ and $F(s)$ are defined for $t\geqslant 0$ only.

PROPERTIES OF LAPLACE TRANSFORM

- **Differentiation**

$$\mathfrak{L}\left\{\frac{df}{dt}\right\}=\int_0^\infty f'\cdot e^{-st}dt=f(t)e^{-st}\Big|_0^\infty-\int_0^\infty f\cdot(-s)e^{-st}dt \qquad (\text{by } i.b.p.)$$
$$=s\int_0^\infty f\cdot e^{-st}dt-f(0)=sF(s)-f(0)$$

$$\mathfrak{L}\left\{\frac{d^2f}{dt^2}\right\}=\int_0^\infty f''\cdot e^{-st}dt=f(t)'e^{-st}\Big|_0^\infty-\int_0^\infty f'\cdot(-s)e^{-st}dt=s\int_0^\infty f'\cdot e^{-st}dt-f'(0)$$
$$=s(sF(s)-f(0))-f'(0)=s^2F(s)-sf(0)-f'(0)$$

$$\vdots$$

$$\mathfrak{L}\left\{\frac{d^nf}{dt^n}\right\}=\int_0^\infty f^{(n)}\cdot e^{-st}dt=f(t)^{(n-1)}e^{-s}\Big|_0^\infty-\int_0^\infty f^{(n-1)}\cdot(-s)e^{-st}dt$$
$$=s\int_0^\infty f^{(n-1)}\cdot e^{-s}\,dt-f^{(n-1)}(0)=s\left(\mathfrak{L}\left\{\frac{d^{n-1}f}{dt^{n-1}}\right\}\right)-f^{(n-1)}(0)$$
$$=s^nF(s)-s^{n-1}f(0)-\cdots-sf^{(n-2)}(0)-f^{(n-1)}(0)$$

- **If** $f(0)=f'(0)=f''(0)=\ldots=f^{(n-1)}(0)=0$,
  - **Initial condition effects are vanished.**
  - **It is very convenient to use deviation variables so that all the effects of initial condition vanish.**

$$\mathfrak{L}\left\{\frac{df}{dt}\right\}=sF(s)$$
$$\mathfrak{L}\left\{\frac{d^2f}{dt^2}\right\}=s^2F(s)$$
$$\vdots$$
$$\mathfrak{L}\left\{\frac{d^nf}{dt^n}\right\}=s^nF(s)$$

- **Transforms of linear differential equations.**

$$y(t)\xrightarrow{\mathfrak{L}}Y(s),\qquad u(t)\xrightarrow{\mathfrak{L}}U(s)$$
$$\frac{dy(t)}{dt}\xrightarrow{\mathfrak{L}}sY(s)\quad(\text{if } y(0)=0)$$

$$\tau\frac{dy(t)}{dt}=-y(t)+Ku(t)\ \ (y(0)=0)\xrightarrow{\mathfrak{L}}(\tau s+1)Y(s)=KU(s)$$

$$\frac{\partial T_L}{\partial t}=-v\frac{\partial T_L}{\partial z}+\frac{1}{\tau_{HL}}(T_w-T_L)\xrightarrow{\mathfrak{L}}\tau_{HL}v\frac{\partial\tilde{T}_L(s)}{\partial z}+(\tau_{HL}s+1)\tilde{T}_L(s)=\tilde{T}_w(s)$$

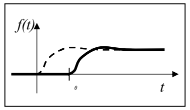
- **Integration**

$$\mathfrak{L}\left\{\int_0^t f(\xi)d\xi\right\} = \int_0^\infty \left(\int_0^t f(\xi)d\xi\right)e^{-st}dt$$
$$= \frac{e^{-s}}{-s}\int_0^t f(\xi)d\xi\Big|_0^\infty + \frac{1}{s}\int_0^\infty f\cdot e^{-st}dt = \frac{F(s)}{s} \qquad \text{(by } i.b.p.\text{)}$$

0

$$\left(\text{Leibniz rule: } \frac{d}{dt}\int_{a(t)}^{b(t)} f(\tau)d\tau = f(b(t))\frac{db(t)}{dt} - f(a(t))\frac{da(t)}{dt}\right)$$

- **Time delay (Translation in time)**



$$f(t) \xrightarrow{+\theta \text{ in} t} f(t-\theta)\,S(t-\theta)$$

$$\mathfrak{L}\{f(t-\theta)\,S(t-\theta)\} = \int_\theta^\infty f(t-\theta)e^{-st}dt = \int_0^\infty f(\tau)e^{-s(\tau+\theta)}d\tau \quad (\text{let } \tau = t-\theta)$$
$$= e^{-\theta}\int_0^\infty f(\tau)e^{-\tau s}d\tau = e^{-\theta}\,F(s)$$

- **Derivative of Laplace transform**

$$\frac{dF(s)}{ds} = \frac{d}{ds}\int_0^\infty f\cdot e^{-s}\,dt = \int_0^\infty f\cdot\frac{d}{ds}e^{-s}\,dt = \int_0^\infty (-t\cdot f)e^{-s}\,dt = \mathfrak{L}[-t\cdot f(t)]$$

- **Final value theorem**
  - **From the LT of differentiation, as s approaches to zero**

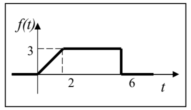
$$\lim_{s\to 0}\int_0^\infty \frac{df}{dt}\cdot e^{-st}dt = \int_0^\infty \frac{df}{dt}\cdot\lim_{s\to 0} e^{-st}\,dt = \lim_{s\to 0}[sF(s) - f(0)]$$
$$\int_0^\infty \frac{df}{dt}dt = f(\infty) - f(0) = \lim_{s\to 0} s\,F(s) - f(0) \Rightarrow \boxed{f(\infty) = \lim_{s\to 0} s\,F(s)}$$

  - **Limitation: $f(\infty)$ has to exist. If it diverges or oscillates, this theorem is not valid.**

- **Initial value theorem**
  - **From the LT of differentiation, as s approaches to infinity**

$$\lim_{s\to\infty}\int_0^\infty \frac{df}{dt}\cdot e^{-st}dt = \lim_{s\to\infty}[sF(s) - f(0)]$$
$$\lim_{s\to\infty}\int_0^\infty \frac{df}{dt}e^{-s}\,dt = 0 = \lim_{s\to\infty} s\,F(s) - f(0) \Rightarrow \boxed{f(0) = \lim_{s\to\infty} s\,F(s)}$$

EXAMPLE ON LAPLACE TRANSFORM (1)

- 

$$f(t) = \begin{cases} 1.5t & \text{for } 0 \le t < 2 \\ 3 & \text{for } 2 \le t < 6 \\ 0 & \text{for } 6 \le t \\ 0 & \text{for } t < 0 \end{cases}$$

$$f(t) = 1.5t\,S(t) - 1.5(t-2)\,S(t-2) - 3\,S(t-6)$$
$$\therefore F(s) = \mathfrak{L}\{f(t)\} = \frac{1.5}{s^2}(1 - e^{-2}\,) - \frac{3}{s}e^{-6s}$$

- For $F(s) = \dfrac{2}{s-5}$, find $f(0)$ and $f(\infty)$.
  - **Using the initial and final value theorems**

$$f(0) = \lim_{s\to\infty} s\,F(s) = \lim_{s\to\infty}\frac{2s}{s-5} = 2 \qquad f(\infty) = \lim_{s\to 0} s\,F(s) = \lim_{s\to 0}\frac{2s}{s-5} = 0$$

  - **But the final value theorem is not valid because**

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty} 2\,e^{5t} = \infty$$

EXAMPLE ON LAPLACE TRANSFORM (2)

- **What is the final value of the following system?**

$$x'' + x' + x = \sin t\,;\ x(0) = x'(0) = 0$$
$$\Rightarrow s^2X(s) + sX(s) + X = \frac{1}{s^2+1} \Rightarrow \mathrm{x}(s) = \frac{1}{(s^2+1)(s^2+s+1)}$$
$$x(\infty) = \lim_{s\to 0}\frac{s}{(s^2+1)(s^2+s+1)} = 0$$

  - **Actually, $x(\infty)$ cannot be defined due to sin t term.**

- **Find the Laplace transform for $(t\sin\omega\,t)$?**

From $\dfrac{dF(s)}{ds} = \mathfrak{L}[-t\cdot f(t)]$

$$\mathfrak{L}[t\cdot\sin\omega\,t] = -\frac{d}{ds}\left[\frac{\omega}{s^2+\omega^2}\right] = \frac{2\omega s}{(s^2+\omega^2)^2}$$

INVERSE LAPLACE TRANSFORM

- **Used to recover the solution in time domain**

$$\boxed{\mathfrak{L}^{-1}\{F(s)\} = f(t)}$$

  - **From the table**
  - **By partial fraction expansion**
  - **By inversion using contour integral**

$$f(t) = \mathfrak{L}^{-1}\{F(s)\} = \frac{1}{2\pi j}\oint_C e^{st}F(s)ds$$

- **Partial fraction expansion**
  - **After the partial fraction expansion, it requires to know some simple formula of inverse Laplace transform such as**

$$\frac{1}{(\tau s+1)}, \frac{s}{(s+b)^2+\omega^2}, \frac{(n-1)!}{s^n}, \frac{e^{-\theta s}}{\tau^2 s^2 + 2\zeta\tau s + 1}, \text{ etc.}$$

PARTIAL FRACTION EXPANSION

$$\boxed{F(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{(s+p_1)\cdots(s+p_n)} = \frac{\alpha_1}{(s+p_1)} + \cdots + \frac{\alpha_n}{(s+p_n)}}$$

- **Case I: All p_i's are distinct and real**
  - **By a root-finding technique, find all roots (time-consuming)**
  - **Find the coefficients for each fraction**
    - **Comparison of the coefficients after multiplying the denominator**
    - **Replace some values for s and solve linear algebraic equation**
    - **Use of Heaviside expansion**
      - **Multiply both side by a factor, $(s+p_i)$, and replace s with $-p_i$.**

$$\alpha_i = (s+p_i)\left.\frac{N(s)}{D(s)}\right|_{s=-p_i}$$

  - **Inverse LT:**

$$f(t) = \alpha_1 e^{-p_1 t} + \alpha_2 e^{-p_2 t} + \cdots + \alpha_n e^{-p_n t}$$

- **Case II: Some roots are repeated**

$$\boxed{F(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{(s+p)^r} = \frac{b_{r-1}s^{r-1} + \cdots + b_0}{(s+p)^r} = \frac{\alpha_1}{(s+p)} + \cdots + \frac{\alpha_r}{(s+p)^r}}$$

  - **Each repeated factors have to be separated first.**
  - **Same methods as Case I can be applied.**
  - **Heaviside expansion for repeated factors**

$$\alpha_{r-i} = \frac{1}{i!}\frac{d^{(i)}}{ds^{(i)}}\left.\left(\frac{N(s)}{D(s)}(s+p)^r\right)\right|_{s=-p} \quad (i = 0, \cdots, r-1)$$

  - **Inverse LT**

$$f(t) = \alpha_1 e^{-p} + \alpha_2 t e^{-pt} + \cdots + \frac{\alpha_r}{(r-1)!}t^{r-1}e^{-pt}$$

- **Case III: Some roots are complex**

$$\boxed{F(s) = \frac{N(s)}{D(s)} = \frac{c_1 s + c_0}{s^2 + d_1 s + d_0} = \frac{\alpha_1(s+b) + \beta_1\omega}{(s+b)^2 + \omega^2}}$$

  - **Each repeated factors have to be separated first.**
  - **Then,**

$$\frac{\alpha_1(s+b) + \beta_1\omega}{(s+b)^2+\omega^2} = \alpha_1\frac{(s+b)}{(s+b)^2+\omega^2} + \beta_1\frac{\omega}{(s+b)^2+\omega^2}$$

where $b = d_1/2, \quad \omega = \sqrt{d_0 - {d_1}^2/4}$

$\alpha_1 = c_1, \quad \beta_1 = (c_0 - \alpha_1 b)/\omega$

  - **Inverse LT**

$$f(t) = \alpha_1 e^{-bt}\cos\omega t + \beta_1 e^{-b}\ \sin\omega t$$

EXAMPLES ON INVERSE LAPLACE TRANSFORM

- $F(s) = \dfrac{(s+5)}{s(s+1)(s+2)(s+3)} = \dfrac{A}{s} + \dfrac{B}{s+1} + \dfrac{C}{s+2} + \dfrac{D}{s+3}$ (distinct)

  - **Multiply each factor and insert the zero value**

$$\left.\frac{(s+5)}{(s+1)(s+2)(s+3)}\right|_{s=0} = \left.\left(A + s\frac{B}{s+1} + s\frac{C}{s+2} + s\frac{D}{s+3}\right)\right|_{s=0} \Rightarrow A = 5/6$$

$$\left.\frac{(s+5)}{s(s+2)(s+3)}\right|_{s=-1} = \left.\left(\frac{A(s+1)}{s} + B + \frac{C(s+1)}{s+2} + \frac{D(s+1)}{s+3}\right)\right|_{s=-1} \Rightarrow B = -2$$

$$\left.\frac{(s+5)}{s(s+1)(s+3)}\right|_{s=-2} = \left.\left(\frac{A(s+2)}{s} + \frac{B(s+2)}{s+1} + C + \frac{D(s+2)}{s+3}\right)\right|_{s=-2} \Rightarrow C = 3/2$$

$$\left.\frac{(s+5)}{s(s+1)(s+2)}\right|_{s=-3} = \left.\left(\frac{A(s+3)}{s} + \frac{B(s+3)}{s+1} + \frac{C(s+3)}{s+2} + D\right)\right|_{s=-3} \Rightarrow D = -1/3$$

$$\therefore f(t) = \mathfrak{L}^{-1}\{F(s)\} = \frac{5}{6} - 2e^{-t} + \frac{3}{2}e^{-2t} - \frac{1}{3}e^{-3}$$

- $F(s) = \dfrac{1}{(s+1)^3(s+2)} = \dfrac{As^2+Bs+C}{(s+1)^3} + \dfrac{D}{(s+2)}$ (repeated)

$$\begin{aligned} 1 &= (As^2+Bs+C)(s+2) + D(s+1)^3 \\ &= (A+D)s^3 + (2A+B+3D)s^2 + (2B+C+3D)s + (2C+D) \end{aligned}$$

$\therefore A = -D, \quad 2A+B+3D = 0, \quad 2B+C+3D = 0, \quad 2C+D = 1$

$\Rightarrow A = 1, \quad B = 1, \quad C = 1, \quad D = -1$

  - **Use of Heaviside expansion** $\alpha_{r-i} = \left.\dfrac{1}{i!}\dfrac{d^{(i)}}{ds^{(i)}}\left(\dfrac{N(s)}{D(s)}(s+p)^r\right)\right|_{s=-p} \quad (i = 0, \cdots, r-1)$

$$\frac{s^2+s+1}{(s+1)^3} = \frac{\alpha_1}{(s+1)} + \frac{\alpha_2}{(s+1)^2} + \frac{\alpha_3}{(s+1)^3}$$

$$(i=0):\ \alpha_3 = \left.(s^2+s+1)\right|_{s=-1} = 1$$

$$(i=1):\ \alpha_2 = \left.\frac{1}{1!}\frac{d}{ds}(s^2+s+1)\right|_{s=-1} = -1$$

$$(i=2):\ \alpha_1 = \left.\frac{1}{2!}\frac{d^2}{ds^2}(s^2+s+1)\right|_{s=-1} = 1$$

$$\therefore f(t) = \mathfrak{L}^{-1}\{F(s)\} = e^{-t} - te^{-t} + \frac{1}{2}t^2e^{-t} - e^{-2t}$$

- $F(s) = \dfrac{(s+1)}{s^2(s^2+4s+5)} = \dfrac{A(s+2)+B}{(s+2)^2+1} + \dfrac{Cs+D}{s^2}$ (complex)

$$\begin{aligned} s+1 &= A(s+2)s^2 + Bs^2 + (Cs+D)(s^2+4s+5) \\ &= (A+C)s^3 + (2A+B+4C+D)s^2 + (5C+4D)s + 5D \end{aligned}$$

$\therefore A = -C, \quad 2A+B+4C+D = 0, \quad 5C+4D = 1, \quad 5D = 1$

$\Rightarrow A = -1/25, \quad B = -7/25, \quad C = 1/25, \quad D = 1/5$

$$\frac{A(s+2)+B}{(s+2)^2+1} = -\frac{1}{25}\frac{(s+2)}{(s+2)^2+1} - \frac{7}{25}\frac{B}{(s+2)^2+1}$$

$$\frac{Cs+D}{s^2} = \frac{1}{25}\frac{1}{s} + \frac{1}{5}\frac{1}{s^2}$$

$$\therefore f(t) = \mathfrak{L}^{-1}\{F(s)\} = -\frac{1}{25}e^{-2}\ \cos t - \frac{7}{25}e^{-2}\ \sin t + \frac{1}{25} + \frac{1}{5}t$$

- $F(s) = \dfrac{1+e^{-2s}}{(4s+1)(3s+1)} = \left(\dfrac{A}{4s+1} + \dfrac{B}{3s+1}\right)(1+e^{-2}\ )$ (Time delay)

$A = 1/(3s+1)\big|_{s=-1/4} = 4, \qquad B = 1/(4s+1)\big|_{s=-1/3} = -3$

$$\begin{aligned} \therefore f(t) = \mathfrak{L}^{-1}\{F(s)\} &= \mathfrak{L}^{-1}\left\{\frac{4}{4s+1} - \frac{3}{3s+1}\right\} + \mathfrak{L}^{-1}\left\{\frac{4e^{-2s}}{4s+1} - \frac{3e^{-2s}}{3s+1}\right\} \\ &= e^{-t/4} - e^{-t/3} + \left(e^{-(t-2)/4} - e^{-(t-2)/3}\right)\mathrm{S}(t-2) \end{aligned}$$

SOLVING ODE BY LAPLACE TRANSFORM

- **Procedure**
  1. **Given linear ODE with initial condition,**
  2. **Take Laplace transform and solve for output**
  3. **Inverse Laplace transform**

- **Example:** $\boxed{\text{Solve for } 5\dfrac{dy}{dt} + 4y = 2;\ y(0) = 1}$

$$\mathfrak{L}\left\{5\frac{dy}{dt}\right\} + \mathfrak{L}\{4y\} = \mathfrak{L}\{2\} \;\Rightarrow\; 5(sY(s) - y(0)) + 4Y(s) = \frac{2}{s}$$

$$(5s+4)Y(s) = \frac{2}{s} + 5 \;\Rightarrow\; Y(s) = \frac{5s+2}{s(5s+4)}$$

$$\therefore y(t) = \mathfrak{L}^{-1}\{Y(s)\} = \mathfrak{L}^{-1}\left\{\frac{0.5}{s} + \frac{2.5}{5s+4}\right\} = 0.5 + 0.5e^{-0.8t}$$

TRANSFER FUNCTION (1)

- **Definition**
  - **An algebraic expression for the dynamic relation between the input and output of the process model**

$$5\frac{dy}{dt} + 4y = u;\ y(0) = 1$$

Let $\tilde{y} = y - 1$ and $\tilde{u} = u - 4$

$$(5s+4)\tilde{Y}(s) = \tilde{U}(s) \;\Rightarrow\; \frac{\tilde{Y}(s)}{\tilde{U}(s)} = \frac{1}{5s+4} = \frac{0.25}{1.25s+1} = G(s)$$

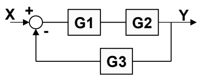
- **How to find transfer function**
  1. **Find the equilibrium point**
  2. **If the system is nonlinear, then linearize around equil. point**
  3. **Introduce deviation variables**
  4. **Take Laplace transform and solve for output**
  5. **Do the Inverse Laplace transform and recover the original variables from deviation variables**

TRANSFER FUNCTION (2)

- **Benefits**
  - **Once TF is known, the output response to various given inputs can be obtained easily.**

$$y(t) = \mathfrak{L}^{-1}\{Y(s)\} = \mathfrak{L}^{-1}\{G(s)U(s)\} \neq \mathfrak{L}^{-1}\{G(s)\}\mathfrak{L}^{-1}\{U(s)\}$$

  - **Interconnected system can be analyzed easily.**
    - **By block diagram algebra**



$$\Rightarrow \quad \frac{Y(s)}{X(s)} = \frac{G1(s)G2(s)}{1 + G1(s)G2(s)G3(s)}$$

  - **Easy to analyze the qualitative behavior of a process, such as stability, speed of response, oscillation, etc.**
    - **By inspecting "Poles" and "Zeros"**
    - **Poles: all s's satisfying $D(s)$=0**
    - **Zeros: all s's satisfying $N(s)$=0**

TRANSFER FUNCTION (3)

- **Steady-state Gain: The ratio between ultimate changes in input and output**

$$\text{Gain}{=}K = \frac{\Delta\text{ouput}}{\Delta\text{input}} = \frac{(y(\infty) - y(0))}{(u(\infty) - u(0))}$$

  - **For a unit step change in input, the gain is the change in output**
  - **Gain may not be definable: for example, integrating processes and processes with sustaining oscillation in output**
  - **From the final value theorem, unit step change in input with zero initial condition gives**

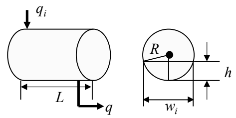
$$K = \frac{y(\infty)}{1} = \lim_{s\to 0} s\,Y(s) = \lim_{s\to 0} s\,G(s)\frac{1}{s} = \lim_{s\to 0} G\,(s)$$

  - **The transfer function itself is an impulse response of the system** $Y(s) = G(s)U(s) = G(s)\mathfrak{L}\{\delta(\mathrm{t})\} = G(s)$

EXAMPLE

- **Horizontal cylindrical storage tank (Ex4.7)**

$$\frac{dm}{dt}=\rho\frac{dV}{dt}=\rho q_i-\rho q$$



$$V(h)=\int_0^h Lw_i(\bar{h})d\bar{h}\Rightarrow\frac{dV}{dt}=Lw_i(h)\frac{dh}{dt}$$

$$w_i(h)/2=\sqrt{R^2-(R-h)^2}=\sqrt{(2R-h)h}$$

$$w_iL\frac{dh}{dt}=q_i-q\ \Rightarrow\ \frac{dh}{dt}=\frac{1}{2L\sqrt{(D-h)h}}(q_i-q)\quad\text{(Nonlinear ODE)}$$

– **Equilibrium point:** $(\bar{q}_i,\bar{q},\bar{h})$ $\qquad 0=(\bar{q}_i-\bar{q})/(2L\sqrt{(D-\bar{h})\bar{h}})$
(if $\bar{q}_i=\bar{q}$, $\bar{h}$ can be any value in $0\le\bar{h}\le D$.)

– **Linearization:**

$$\frac{dh}{dt}=f(h,q_i,q)=\left.\frac{\partial f}{\partial h}\right|_{(\bar{h},\bar{q}_i,\bar{q})}(h-\bar{h})+\left.\frac{\partial f}{\partial q_i}\right|_{(\bar{h},\bar{q}_i,\bar{q})}(q_i-\bar{q}_i)+\left.\frac{\partial f}{\partial q}\right|_{(\bar{h},\bar{q}_i,\bar{q})}(q-\bar{q})$$

$$\left.\frac{\partial f}{\partial h}\right|_{(\bar{h},\bar{q}_i,\bar{q})}=(\bar{q}_i-\bar{q})\frac{\partial}{\partial h}\frac{-1}{2L\sqrt{(D-h)h}}=0\quad(\because\bar{q}_i=\bar{q})$$

$$\left.\frac{\partial f}{\partial q}\right|_{(\bar{h},\bar{q}_i,\bar{q})}=\frac{-1}{2L\sqrt{(D-\bar{h})\bar{h}}},\qquad\left.\frac{\partial f}{\partial q_i}\right|_{(\bar{h},\bar{q}_i,\bar{q})}=\frac{1}{2L\sqrt{(D-\bar{h})\bar{h}}}$$

Let this term be k

$$s\tilde{H}(s)=k\tilde{Q}_i(s)-k\tilde{Q}(s)$$

- **Transfer function between** $\boxed{\tilde{H}(s)\text{ and }\tilde{Q}(s):\ -\frac{k}{s}}$ **(integrating)**
- **Transfer function between** $\boxed{\tilde{H}(s)\text{ and }\tilde{Q}_i(s):\ \frac{k}{s}}$ **(integrating)**

- **If $\bar{h}$ is near 0 or D, k becomes very large and $\bar{h}$ is around D/2, k becomes minimum.**

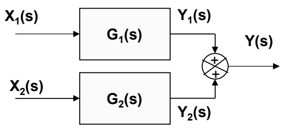
⇒ **The model could be quite different depending on the operating condition used for the linearization.**

⇒ **The best suitable range for the linearization in this case is around D/2. (less change in gain)**

⇒ **Linearized model would be valid in very narrow range near 0.**

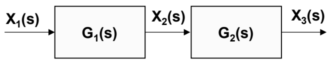
PROPERTIES OF TRANSFER FUNCTION

- **Additive property**



$$\begin{aligned}Y(s)&=Y_1(s)+Y_2(s)\\&=G_1(s)X_1(s)+G_2(s)X_2(s)\end{aligned}$$

- **Multiplicative property**



$$\begin{aligned}X_3(s)&=G_2(s)X_2(s)\\&=G_2(s)[G_1(s)X_1(s)]\end{aligned}$$

- **Physical realizability**
  - **In a transfer function, the order of numerator(m) is greater than that of denominator(n): called "physically unrealizable"**
  - **The order of derivative for the input is higher than that of output. (requires future input values for current output)**

EXAMPLES ON TWO TANK SYSTEM

- **Two tanks in series (Ex3.7)**
  - **No reaction**

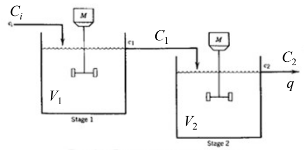


Figure 3.4. Two-stage stirred-tank reactor system.

$$V_1\frac{dc_1}{dt}+qc_1=qc_i$$

$$V_2\frac{dc_2}{dt}+qc_2=qc_1$$

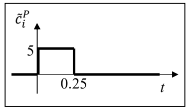
  - **Initial condition:** $c_1(0)=c_2(0)=1$ kg mol/m^3 **(Use deviation var.)**
  - **Parameters:** V_1/q**=2 min.,** V_2/q**=1.5 min.**
  - **Transfer functions**

$$\frac{\tilde{C}_1(s)}{\tilde{C}_i(s)}=\frac{1}{(V_1/q)s+1}\qquad\frac{\tilde{C}_2(s)}{\tilde{C}_1(s)}=\frac{1}{(V_2/q)s+1}$$

$$\frac{\tilde{C}_2(s)}{\tilde{C}_i(s)}=\frac{\tilde{C}_2(s)}{\tilde{C}_1(s)}\frac{\tilde{C}_1(s)}{\tilde{C}_i(s)}=\frac{1}{((V_2/q)s+1)((V_1/q)s+1)}$$

- **Pulse input**

$$\tilde{C}_i^P(s) = \frac{5}{s}(1 - e^{-0.25s})$$



- **Equivalent impulse input**

$$\tilde{C}_i^\delta(s) = \mathfrak{L}\{(5 \times 0.25)\delta(t)\} = 1.25$$

- **Pulse response vs. Impulse response**

$$\tilde{C}_1^P(s) = \frac{1}{2s+1}\tilde{C}_i^P(s) = \frac{5}{s(2s+1)}(1 - e^{-0.25s})$$
$$= \left(\frac{5}{s} - \frac{10}{2s+1}\right)(1 - e^{-0.25s})$$
$$\Rightarrow \boxed{\begin{aligned}\tilde{c}_1^P(t) &= 5(1 - e^{-t/2}) \\ &\quad - 5(1 - e^{-(t-0.25)/2})\,S(t - 0.25)\end{aligned}}$$

$$\tilde{C}_1^\delta(s) = \frac{1}{2s+1}\tilde{C}_i^\delta(s) = \frac{1.25}{(2s+1)}$$
$$\Rightarrow \boxed{\tilde{c}_1^\delta = 0.625e^{-t/2}}$$

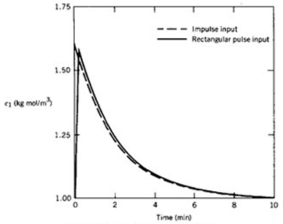


Figure 3.6. Reactor Stage 1 response.

$$\tilde{C}_2^P(s) = \frac{1}{(2s+1)(1.5s+1)}\tilde{C}_i^P(s) = \frac{5}{s(2s+1)(1.5s+1)}(1 - e^{-0.25s})$$
$$= \left(\frac{5}{s} - \frac{40}{2s+1} + \frac{22.5}{1.5s+1}\right)(1 - e^{-0.25s})$$

$$\Rightarrow \boxed{\begin{aligned}\tilde{c}_2^P(t) &= (5 - 20e^{-t/2} + 15e^{-t/1.5}) \\ &\quad - (5 - 20e^{-(t-0.25)/2} + 15e^{-(t-0.25)/1.5})\,S(t - 0.25)\end{aligned}}$$

$$\tilde{C}_2^\delta(s) = \frac{1}{(2s+1)(1.5s+1)}\tilde{C}_i^\delta(s)$$
$$= \frac{1.25}{(2s+1)(1.5s+1)}$$
$$= \frac{5}{2s+1} - \frac{3.75}{1.5s+1}$$
$$\Rightarrow \boxed{\tilde{c}_2^\delta = 2.5e^{-t/2} - 2.5e^{-t/1.5}}$$

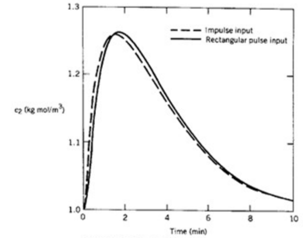


Figure 3.7. Reactor Stage 2 response.