CHBE320 LECTURE IX
FREQUENCY RESPONSES

Professor Dae Ryook Yang

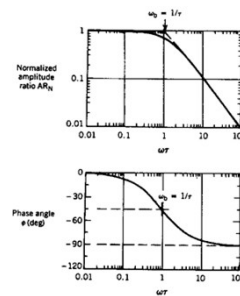### Fall 2021
Dept. of Chemical and Biological Engineering
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Road Map of the Lecture IX

- **Frequency Response**
  - Definition
  - Benefits of frequency analysis
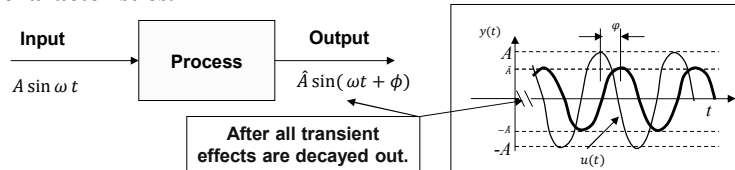 - How to get frequency response
  - Bode Plot
  - Nyquist Diagram

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DEFINITION OF FREQUENCY RESPONSE

- **For linear system**
  - "The ultimate output response of a process for a sinusoidal input at a frequency will show amplitude change and phase shift at the same frequency depending on the process characteristics."
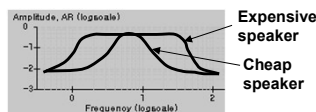

  - **Amplitude ratio (AR): attenuation of amplitude,** $\hat{A}/A$
  - **Phase angle (ϕ): phase shift compared to input**
  - **These two quantities are the function of frequency.**
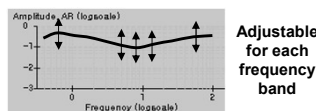
BENEFITS OF FREQUENCY RESPONSE

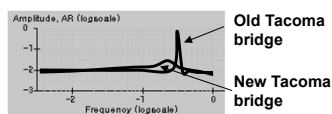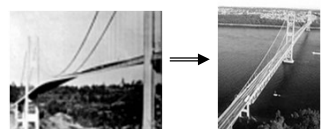- **Frequency responses are the informative representations of dynamic systems**
  - **Audio Speaker**

  - **Equalizer**
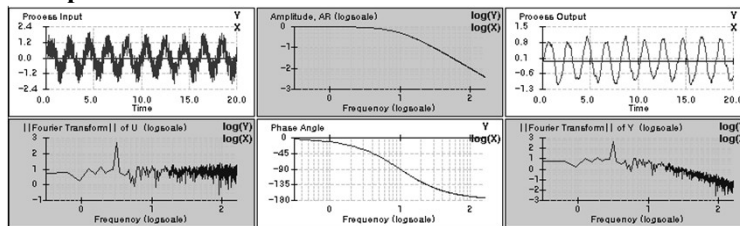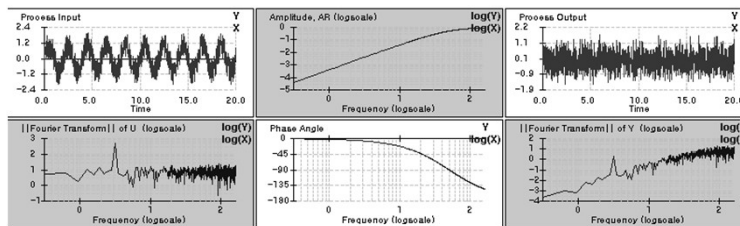

  - **Structure**

  Old Tacoma bridge   New Tacoma bridge

– **Low-pass filter**

– **High-pass filter**

– **In signal processing field, transfer functions are called "filters".**

• **Any linear dynamical system is completely defined by its frequency response.**
  – **The AR and phase angle define the system completely.**
  – **Bode diagram**
    • **AR in log-log plot**
    • **Phase angle in log-linear plot**
  – **Via efficient numerical technique (fast Fourier transform, FFT), the output can be calculated for any type of input.**

• **Frequency response representation of a system dynamics is very convenient for designing a feedback controller and analyzing a closed-loop system.**
  – **Bode stability**
  – **Gain margin (GM) and phase margin (PM)**

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- **Critical frequency**
 - As frequency changes, the amplitude ratio (AR) and the phase angle (PA) change.
  - The frequency where the PA reaches –180° is called critical frequency (ω_c).
  - The component of output at the critical frequency will have the exactly same phase as the signal goes through the loop due to comparator (-180 °) and phase shift of the process (-180 °).
  - For the open-loop gain at the critical frequency, $K_{OL}(\omega_c) = 1$
    - No change in magnitude
    - Continuous cycling
  - For $K_{OL}(\omega_c) > 1$
    - Getting bigger in magnitude
    - Unstable
  - For $K_{OL}(\omega_c) < 1$
    - Getting smaller in magnitude
    - Stable
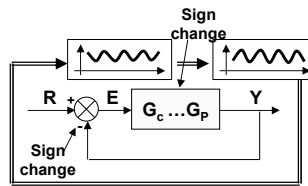
- **Example**
  - If a feed is pumped by a peristaltic pump to a CSTR, will the fluctuation of the feed flow appear in the output?
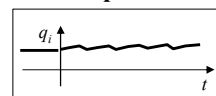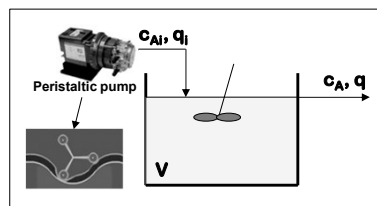
$$V \frac{dc_A}{dt} = q_i c_{Ai} - q c_A \ (q \approx \text{constant})$$

$$\frac{C_A(s)}{q_i(s)} = \frac{C_{Ai}}{Vs + q} = \frac{C_{Ai}/q}{(V/q)s + 1}$$

  - V=50cm³, q=90cm³/min (so is the average of q_i)
    - Process time constant=0.555min.
  - The rpm of the peristaltic pump is 60rpm.
    - Input frequency=180rad/min (3blades)
  - The AR=0.01   ($\omega\tau = 100$)

    If the magnitude of fluctuation of q_i is 5% of nominal flow rate, the fluctuation in the output concentration will be about 0.05% which is almost unnoticeable.
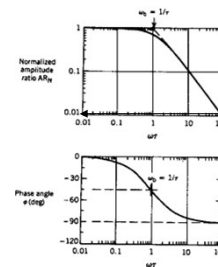
OBTAINING FREQUENCY RESPONSE

- **From the transfer function, replace s with $j\omega$**

$$G(s) \xrightarrow{s=j\omega} G(j\omega)$$

  Transfer function **Frequency response**

  – For a pole, $s = \alpha + j\omega$, the response mode is $e^{(\alpha+j\omega)t}$.
  – If the modes are not unstable ($\alpha \le 0$) and enough time elapses, the survived modes becomes $e^{j\omega t}$. (ultimate response)
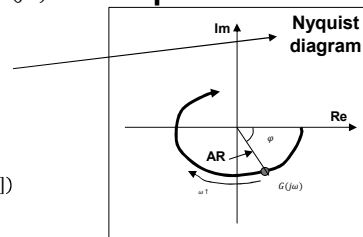
- **The frequency response, $G(j\omega)$ is complex as a function of frequency.**

$$G(j\omega) = \mathrm{Re}[\,G(j\omega)\,] + j\,\mathrm{Im}[\,G(j\omega)\,]$$

$$AR = |G(j\omega)| = \sqrt{\mathrm{Re}[\,G(j\omega)\,]^2 + \mathrm{Im}[\,G(j\omega)\,]^2}$$

$$\phi = \angle G(j\omega) = \tan^{-1}(\mathrm{Im}[\,G(j\omega)\,]/\,\mathrm{Re}[\,G(j\omega)\,])$$

Bode plot

Nyquist diagram

- **Getting ultimate response**
  – For a sinusoidal forcing function $\quad Y(s) = G(s)\dfrac{A\omega}{s^2 + \omega^2}$
  – Assume $G(s)$ has stable poles b_i.   **Decayed out at large t**

$$Y(s) = G(s)\frac{A\omega}{s^2 + \omega^2} = \frac{\alpha_1}{s + b_1} + \cdots + \frac{\alpha_n}{s + b_n} + \frac{Cs + D\omega}{s^2 + \omega^2}$$

$$G(j\omega)A\omega = Cj\omega + D\omega \Rightarrow G(j\omega) = \frac{D}{A} + j\frac{C}{A} = R + jI$$

$$C = IA, D = RA \;\Rightarrow\; y_{ul} = A(I\cos\omega t + R\sin\omega t) = \hat{A}\sin(\omega t + \phi)$$

$$\therefore AR = \hat{A}/A = \sqrt{R^2 + I^2} = |G(j\omega)| \quad \textbf{and} \quad \phi = \tan^{-1}(I/R) = \angle G(j\omega)$$

  – Without calculating transient response, the frequency response can be obtained directly from $G(j\omega)$.
  – Unstable transfer function does not have a frequency response because a sinusoidal input produces an unstable output response.
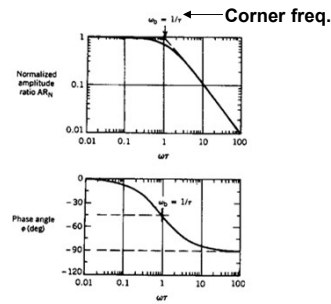
- **First-order process**

$$G(s) = \frac{K}{(\tau s + 1)}$$

$$G(j\omega) = \frac{K}{(1 + j\omega\tau)} = \frac{K}{(1 + \omega^2\tau^2)}(1 - j\omega\tau)$$

$$AR_N = |G(j\omega)| = \frac{1}{\sqrt{1 + \omega^2\tau^2}}$$

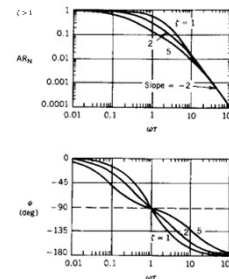$$\phi = \angle G(j\omega) = -\tan^{-1}(\omega\tau)$$

- **Second-order process**

$$G(s) = \frac{K}{(\tau^2 s^2 + 2\zeta\tau s + 1)}$$

$$G(j\omega) = \frac{K}{(1 - \tau^2\omega^2) + 2j\zeta\tau\omega}$$

$$AR = |G(j\omega)| = \frac{K}{\sqrt{(1 - \omega^2\tau^2)^2 + (2\zeta\omega\tau)^2}}$$

$$\phi = \angle G(j\omega) = \tan^{-1}\frac{\mathrm{Im}(G(j\omega))}{\mathrm{Re}(G(j\omega))} = -\tan^{-1}\frac{2\zeta\omega\tau}{1 - \omega^2\tau^2}$$

Corner freq.

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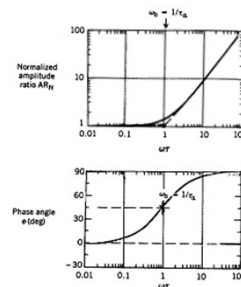
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- **Process Zero (lead)**

$$G(s) = \tau_a s + 1$$

$$G(j\omega) = 1 + j\omega\tau_a$$

$$AR_N = |G(j\omega)| = \sqrt{1 + \omega^2\tau_a^2}$$

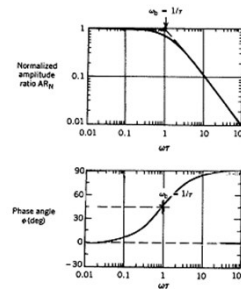$$\phi = \angle G(j\omega) = \tan^{-1}(\omega\tau_a)$$

- **Unstable pole**

$$G(s) = \frac{1}{(-\tau s + 1)}$$

$$G(j\omega) = \frac{1}{1 - j\tau\omega} = \frac{1}{1 + \tau^2\omega^2}(1 + j\tau\omega)$$

$$AR = |G(j\omega)| = \frac{1}{\sqrt{1 + \omega^2\tau^2}}$$

$$\phi = \angle G(j\omega) = \tan^{-1}\frac{\mathrm{Im}(G(j\omega))}{\mathrm{Re}(G(j\omega))} = \tan^{-1}\omega\tau$$

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- **Integrating process**

$$G(s) = \frac{1}{As} \qquad G(j\omega) = \frac{1}{jA\omega} = -\frac{1}{A\omega}j$$

$$AR_N = |G(j\omega)| = \frac{1}{A\omega}$$

$$\phi = \angle G(j\omega) = \tan^{-1}\left(-\frac{1}{0 \cdot \omega}\right) = -\frac{\pi}{2}$$

- **Differentiator**

$$G(s) = As \qquad G(j\omega) = jA\omega$$

$$AR_N = |G(j\omega)| = A\omega$$

$$\phi = \angle G(j\omega) = \tan^{-1}\left(\frac{1}{0 \cdot \omega}\right) = \frac{\pi}{2}$$
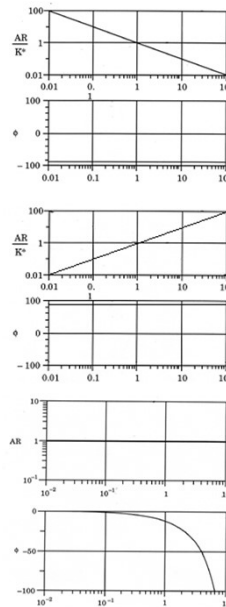
- **Pure delay process**

$$G(s) = e^{-\theta}$$

$$G(j\omega) = e^{-j\theta\omega} = \cos\theta\,\omega - j\sin\theta\,\omega$$

$$AR = |G(j\omega)| = 1$$

$$\phi = \angle G(j\omega) = -\tan^{-1}\tan\theta\,\omega = -\theta\omega$$

SKETCHING BODE PLOT

$$G(s) = \frac{G_a(s)G_b(s)G_c(s)\cdots}{G_1(s)G_2(s)G_3(s)\cdots} \qquad\qquad G(j\omega) = \frac{G_a(j\omega)G_b(j\omega)G_c(j\omega)\cdots}{G_1(j\omega)G_2(j\omega)G_3(j\omega)\cdots}$$

$$|G(j\omega)| = \frac{|G_a(j\omega)||G_b(j\omega)||G_c(j\omega)|\cdots}{|G_1(j\omega)||G_2(j\omega)||G_3(j\omega)|\cdots}$$

$$\angle G(j\omega) = \angle G_a(j\omega) + \angle G_b(j\omega) + \angle G_c(j\omega) + \cdots$$
$$-\angle G_1(j\omega) - \angle G_2(j\omega) - \angle G_3(j\omega) - \cdots$$
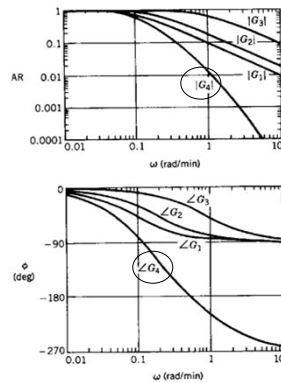
- **Bode diagram**
  - **AR vs. frequency in log-log plot**
  - **PA vs. frequency in semi-log plot**
  - **Useful for**
    - **Analysis of the response characteristics**
    - **Stability of the closed-loop system only for open-loop stable systems with phase angle curves exhibit a single critical frequency.**
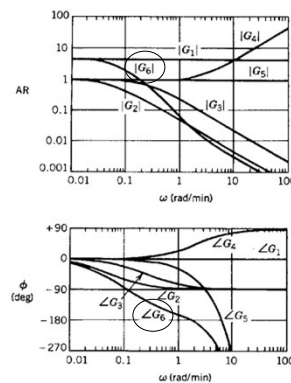
- **Amplitude Ratio on log-log plot**
    - Start from steady-state gain at $\omega = 0$. If G_{OL} includes either integrator or differentiator it starts at ∞ or 0.
    - Each first-order lag (lead) adds to the slope –1 (+1) starting at the corner frequency.
    - Each integrator (differentiator) adds to the slope –1 (+1) starting at zero frequency.
    - A delays does not contribute to the AR plot.
- **Phase angle on semi-log plot**
    - Start from 0° or -180° at $\omega = 0$ depending on the sign of steady-state gain.
    - Each first-order lag (lead) adds 0° to phase angle at $\omega = 0$, adds -90° (+90°) to phase angle at $\omega = \infty$, and adds -45° (+45°) to phase angle at corner frequency.
    - Each integrator (differentiator) adds -90° (+90°) to the phase angle for all frequency.
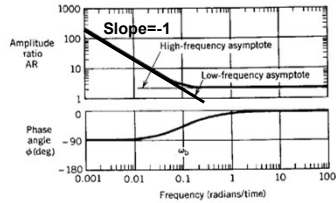   - A delay adds $-\theta\omega$ to phase angle depending on the frequency.

Examples

1. $G(s) = \dfrac{K}{\underset{G_1}{(10s+1)}\,\underset{G_2}{(5s+1)}\,\underset{G_3}{(s+1)}}$

2. $G(s) = \dfrac{\overset{G_1}{5(0.5s+1)}\,\overset{G_4}{}\,\overset{G_5}{e^{-0.5s}}}{\underset{G_2}{(20s+1)}\,\underset{G_3}{(4s+1)}}$

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3. PI: $G(s) = K_C\left(1 + \dfrac{1}{\tau_I s}\right)$ **5. PID:** $G(s) = K_C\left(1 + \dfrac{1}{\tau_I s} + \tau_D s\right)$

$\omega_b = 1/\tau_I$ **at** $\phi = -45°$
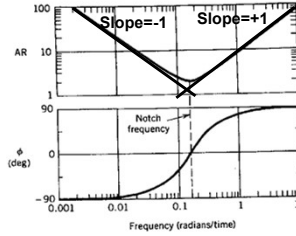
$\omega_{Notch} = \dfrac{1}{\sqrt{\tau_I \tau_D}}$ **at** $\phi = 0°$

4. PD: $G(s) = K_C(1 + \tau_D s)$
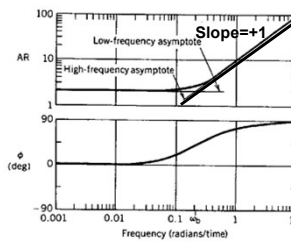
$\omega_b = 1/\tau_D$ **at** $\phi = 45°$

NYQUIST DIAGRAM

- **Alternative representation of frequency response**
- **Polar plot of** $G(j\omega)$ **(ω is implicit)**

$G(j\omega) = \text{Re}[\,G(j\omega)\,] + j\,\text{Im}[\,G(j\omega)\,]$

- – **Compact (one plot)**
- – **Wider applicability of stability analysis than Bode plot**
- – **High frequency characteristics will be shrunk near the origin.**
  - • **Inverse Nyquist diagram: polar plot of** $1/G(j\omega)$
- – **Combination of different transfer function components is not easy as with Nyquist diagram as with Bode plot.**