CHBE320 LECTURE IX
FREQUENCY RESPONSES

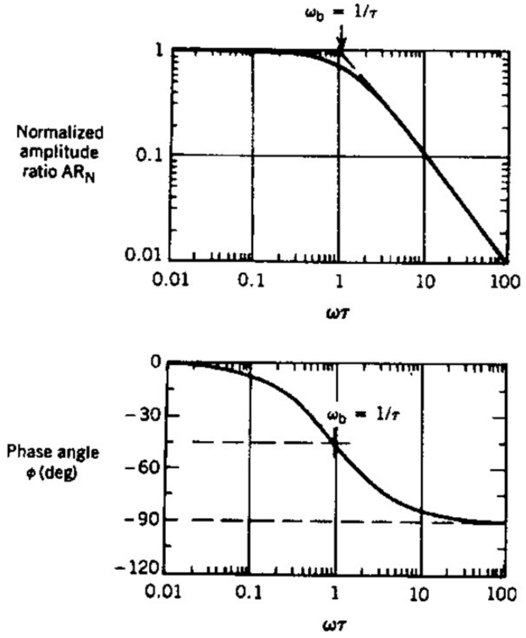
Professor Dae Ryook Yang

Fall 2021

Dept. of Chemical and Biological Engineering
Korea University

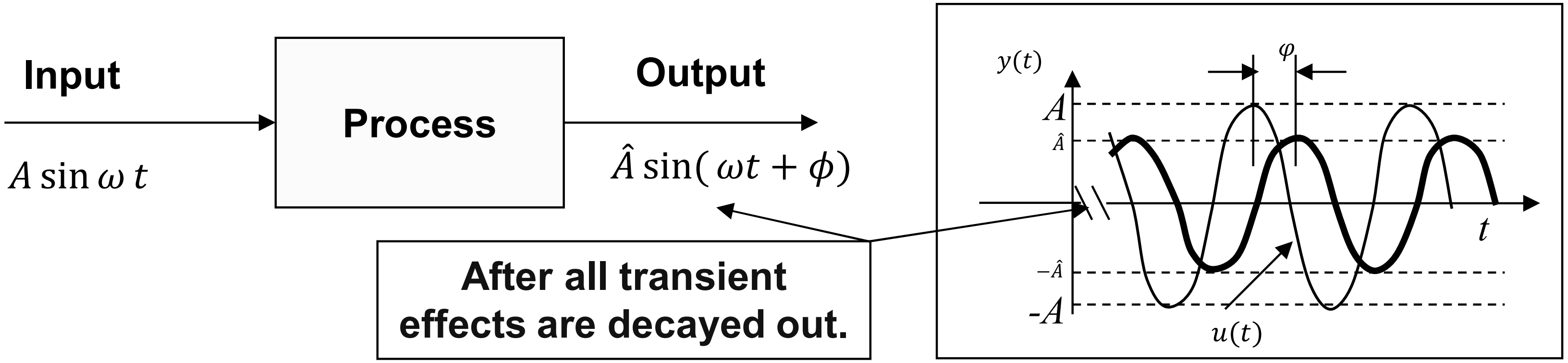
Road Map of the Lecture IX

- **Frequency Response**
  - **Definition**
  - **Benefits of frequency analysis**
  - **How to get frequency response**
  - **Bode Plot**
  - **Nyquist Diagram**



DEFINITION OF FREQUENCY RESPONSE

- **For linear system**
  - **"The ultimate output response of a process for a sinusoidal input at a frequency will show amplitude change and phase shift at the same frequency depending on the process characteristics."**



Input

$A \sin \omega t$

Process

Output

$\hat{A} \sin(\omega t + \phi)$

After all transient effects are decayed out.

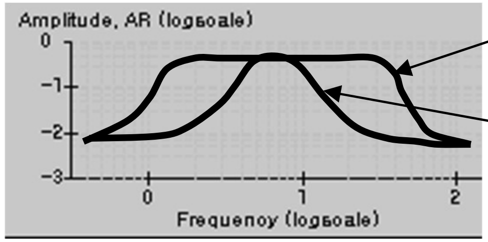
  - **Amplitude ratio (AR): attenuation of amplitude,** $\hat{A}/A$
  - **Phase angle (ϕ): phase shift compared to input**
  - **These two quantities are the function of frequency.**

BENEFITS OF FREQUENCY RESPONSE

- **Frequency responses are the informative representations of dynamic systems**
  - Audio Speaker

Elec. Signal

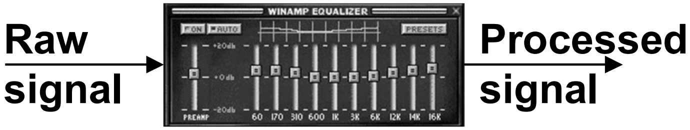
Sound wave



Expensive speaker

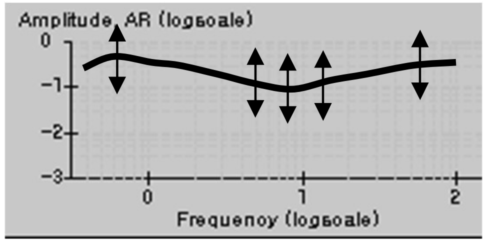
Cheap speaker

  - Equalizer



Raw signal

Processed signal

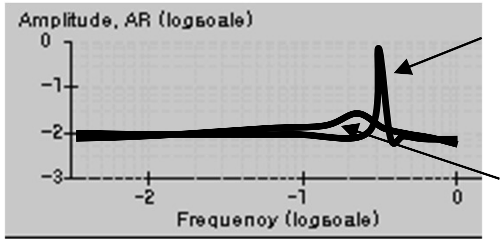


Adjustable for each frequency band

  - Structure

Old Tacoma bridge

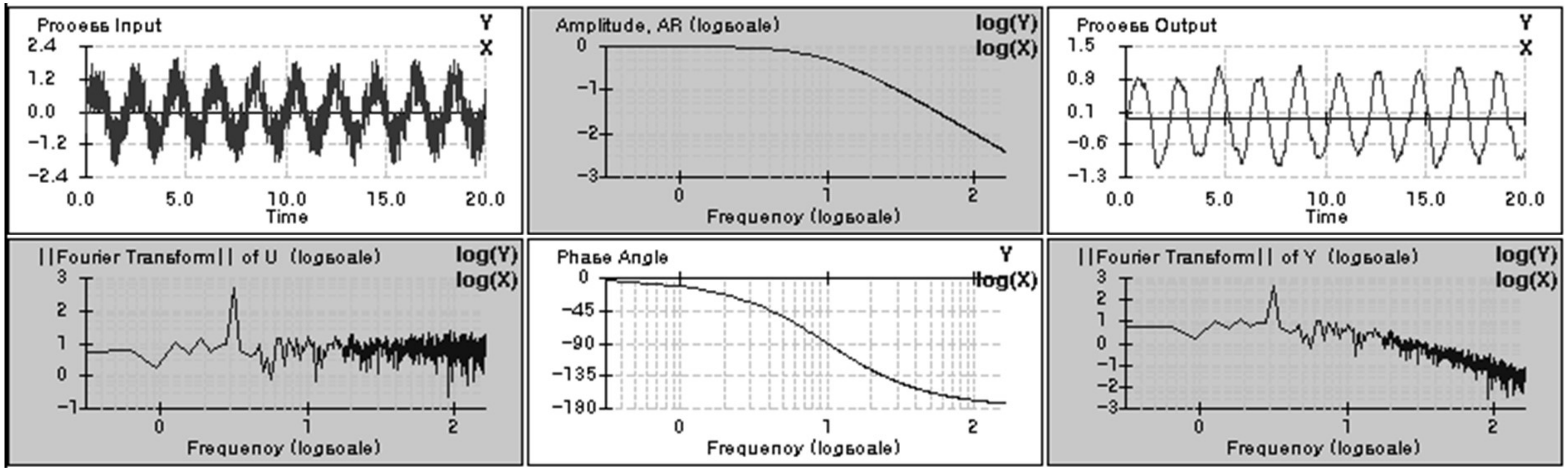
New Tacoma bridge



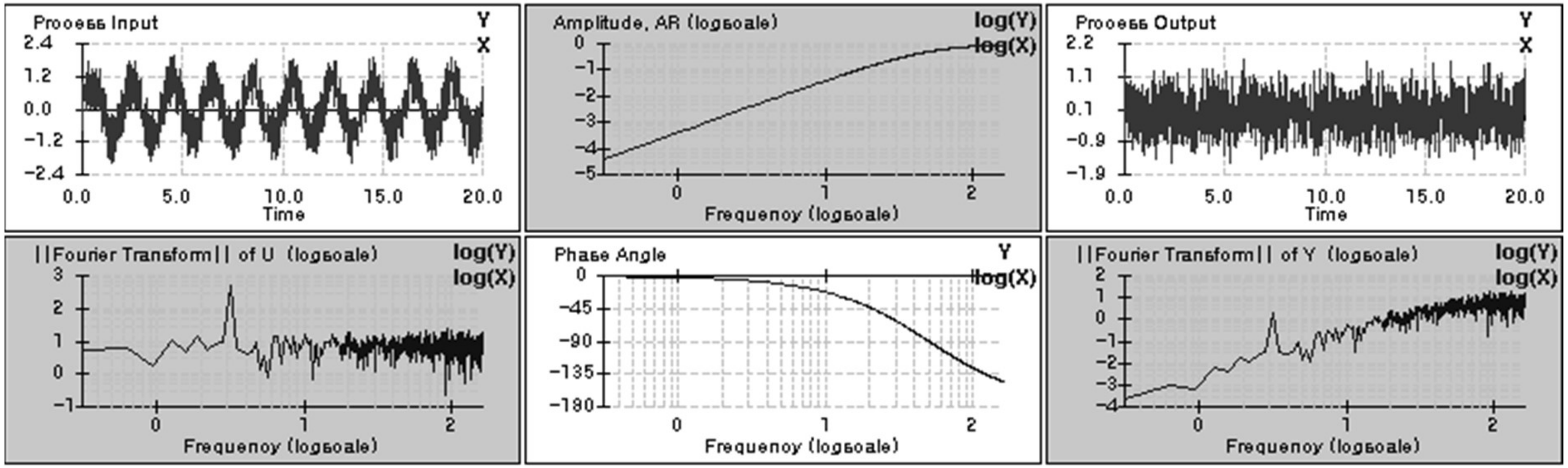
Old Tacoma bridge

New Tacoma bridge

- Low-pass filter



- High-pass filter

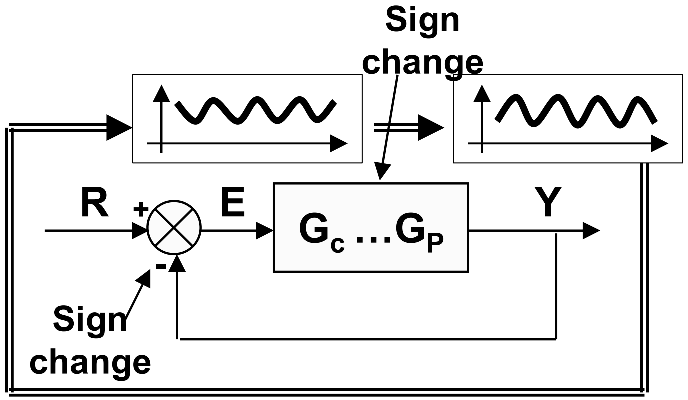


- In signal processing field, transfer functions are called “filters”.

- **Any linear dynamical system is completely defined by its frequency response.**
  - The AR and phase angle define the system completely.
  - Bode diagram
    - AR in log-log plot
    - Phase angle in log-linear plot
  - Via efficient numerical technique (fast Fourier transform, FFT), the output can be calculated for any type of input.

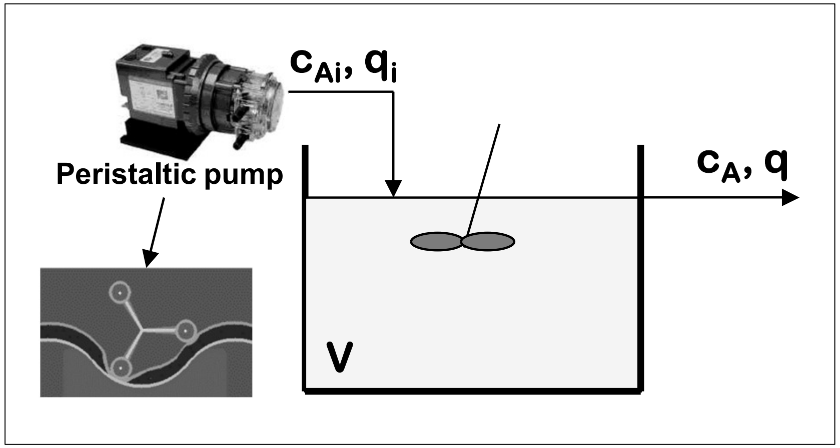
- **Frequency response representation of a system dynamics is very convenient for designing a feedback controller and analyzing a closed-loop system.**
  - Bode stability
  - Gain margin (GM) and phase margin (PM)

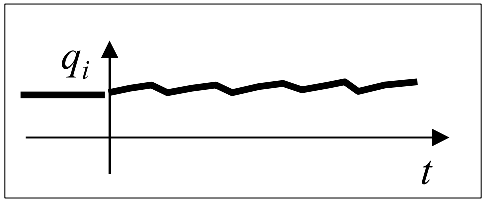
- **Critical frequency**
  - **As frequency changes, the amplitude ratio (AR) and the phase angle (PA) change.**
  - **The frequency where the PA reaches –180° is called critical frequency (ω_c).**
  - **The component of output at the critical frequency will have the exactly same phase as the signal goes through the loop due to comparator (-180 °) and phase shift of the process (-180 °).**
  - **For the open-loop gain at the critical frequency,** $K_{OL}(\omega_c) = 1$
    - **No change in magnitude**
    - **Continuous cycling**
  - **For** $K_{OL}(\omega_c) > 1$
    - **Getting bigger in magnitude**
    - **Unstable**
  - **For** $K_{OL}(\omega_c) < 1$
    - **Getting smaller in magnitude**
    - **Stable**



Example

- If a feed is pumped by a peristaltic pump to a CSTR, will the fluctuation of the feed flow appear in the output?



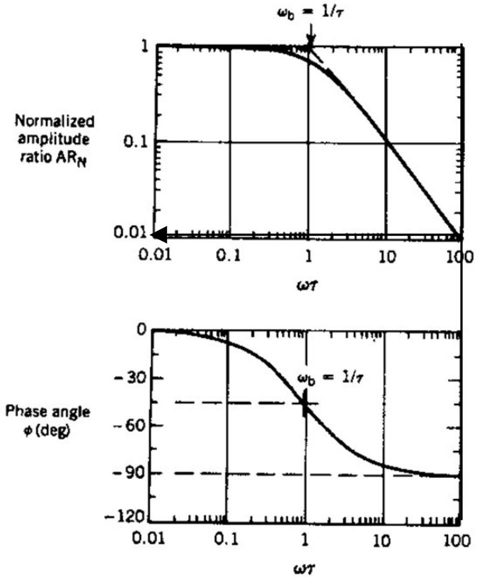


$$V\frac{dc_A}{dt} = q_i c_{Ai} - q c_A \ (q \approx \text{constant})$$

$$\frac{C_A(s)}{q_i(s)} = \frac{C_{Ai}}{Vs+q} = \frac{C_{Ai}/q}{(V/q)s+1}$$

- V=50cm^3, q=90cm^3/min (so is the average of q_i)
  - Process time constant=0.555min.
- The rpm of the peristaltic pump is 60rpm.
  - Input frequency=180rad/min (3blades)
- The AR=0.01 $(\omega\tau = 100)$

If the magnitude of fluctuation of q_i is 5% of nominal flow rate, the fluctuation in the output concentration will be about 0.05% which is almost unnoticeable.



OBTAINING FREQUENCY RESPONSE

- **From the transfer function, replace s with $j\omega$**

$$G(s) \xrightarrow{s=j\omega} G(j\omega)$$

Transfer function

Frequency response

  - **For a pole, $s = \alpha + j\omega$, the response mode is $e^{(\alpha+j\omega)t}$.**
  - **If the modes are not unstable ($\alpha \leq 0$) and enough time elapses, the survived modes becomes $e^{j\omega t}$. (ultimate response)**

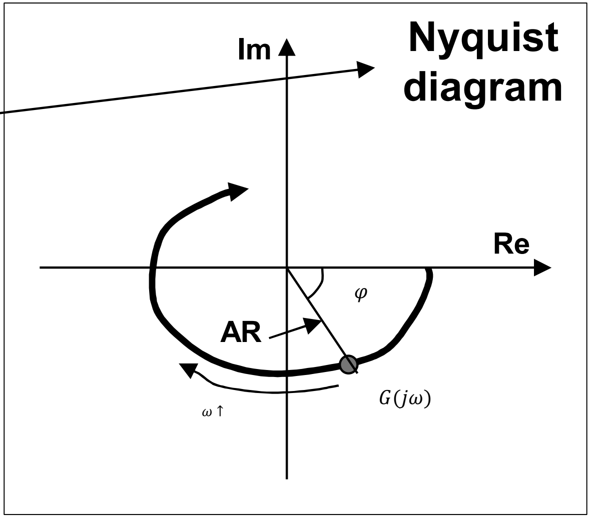
- **The frequency response, $G(j\omega)$ is complex as a function of frequency.**

$$G(j\omega) = \mathrm{Re}[\,G(j\omega)] + j\,\mathrm{Im}[\,G(j\omega)]$$

$$AR = |G(j\omega)| = \sqrt{\mathrm{Re}[\,G(j\omega)]^2 + \mathrm{Im}[\,G(j\omega)]^2}$$

$$\phi = \measuredangle G(j\omega) = \tan^{-1}(\mathrm{Im}[\,G(j\omega)]/\,\mathrm{Re}[\,G(j\omega)])$$

Bode plot



- **Getting ultimate response**
  - **For a sinusoidal forcing function** $Y(s) = G(s)\frac{A\omega}{s^2+\omega^2}$
  - **Assume $G(s)$ has stable poles b_i.**

Decayed out at large *t*

$$Y(s) = G(s)\frac{A\omega}{s^2+\omega^2} = \frac{\alpha_1}{s+b_1} + \cdots + \frac{\alpha_n}{s+b_n} + \frac{Cs+D\omega}{s^2+\omega^2}$$

$$G(j\omega)A\omega = Cj\omega + D\omega \Rightarrow G(j\omega) = \frac{D}{A} + j\frac{C}{A} = R + jI$$

$$C = IA, D = RA \;\Rightarrow\; y_{ul} = A(I\cos\omega\, t + R\sin\omega\, t) = \hat{A}\sin(\,\omega t + \phi)$$

$$\therefore AR = \hat{A}/A = \sqrt{R^2+I^2} = |G(j\omega)| \qquad \text{and} \qquad \phi = \tan^{-1}(\,I/R) = \measuredangle G(j\omega)$$

  - **Without calculating transient response, the frequency response can be obtained directly from $G(j\omega)$.**
  - **Unstable transfer function does not have a frequency response because a sinusoidal input produces an unstable output response.**

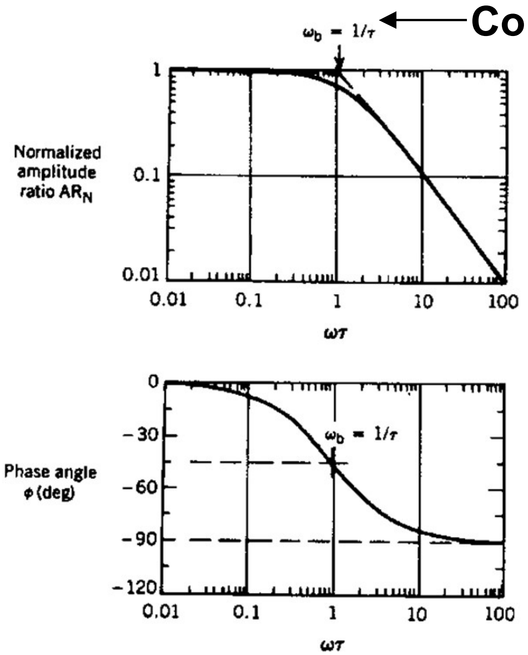
- ## First-order process

$$G(s) = \frac{K}{(\tau s + 1)}$$

$$G(j\omega) = \frac{K}{(1 + j\omega\tau)} = \frac{K}{(1 + \omega^2\tau^2)}(1 - j\omega\tau)$$

$$AR_N = |G(j\omega)| = \frac{1}{\sqrt{1 + \omega^2\tau^2}}$$

$$\phi = \measuredangle G(j\omega) = -\tan^{-1}(\,\omega\tau)$$



Corner freq.

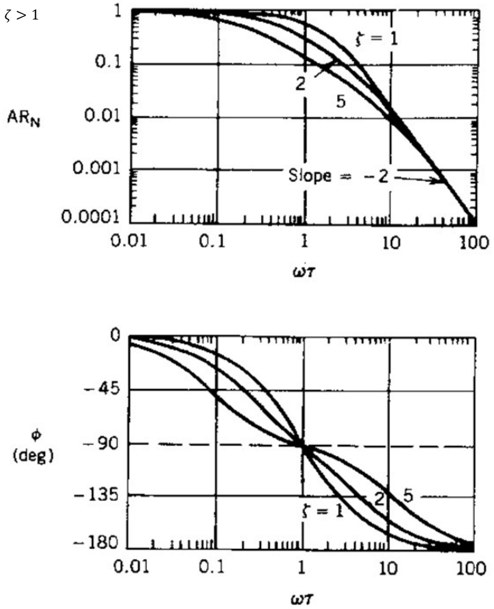
- ## Second-order process

$$G(s) = \frac{K}{(\tau^2 s^2 + 2\zeta\tau s + 1)}$$

$$G(j\omega) = \frac{K}{(1 - \tau^2\omega^2) + 2j\zeta\tau\omega}$$

$$AR = |G(j\omega)| = \frac{K}{\sqrt{(1 - \omega^2\tau^2)^2 + (2\zeta\omega\tau)^2}}$$

$$\phi = \measuredangle G(j\omega) = \tan^{-1}\frac{\mathrm{Im}(\,G(j\omega))}{\mathrm{Re}(\,G(j\omega))} = -\tan^{-1}\frac{2\zeta\omega\tau}{1 - \omega^2\tau^2}$$



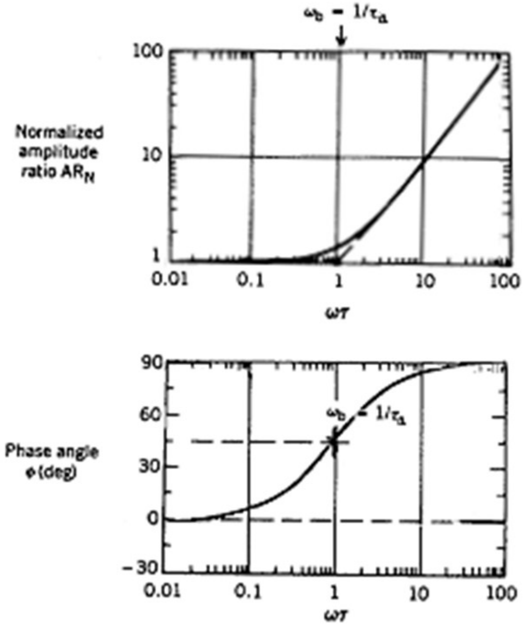
- **Process Zero (lead)**

$$G(s) = \tau_a s + 1$$

$$G(j\omega) = 1 + j\omega\tau_a$$

$$AR_N = |G(j\omega)| = \sqrt{1 + \omega^2\tau_a^2}$$

$$\phi = ∡G(j\omega) = \tan^{-1}(\,\omega\tau_a)$$



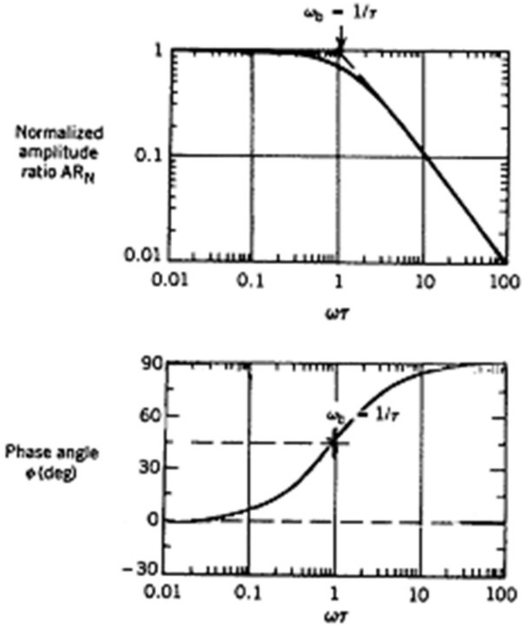
- **Unstable pole**

$$G(s) = \frac{1}{(-\tau s + 1)}$$

$$G(j\omega) = \frac{1}{1 - j\tau\omega} = \frac{1}{1 + \tau^2\omega^2}(1 + j\tau\omega)$$

$$AR = |G(j\omega)| = \frac{1}{\sqrt{1 + \omega^2\tau^2}}$$

$$\phi = ∡G(j\omega) = \tan^{-1}\frac{\mathrm{Im}(\,G(j\omega))}{\mathrm{Re}(\,G(j\omega))} = \tan^{-1}\omega\,\tau$$

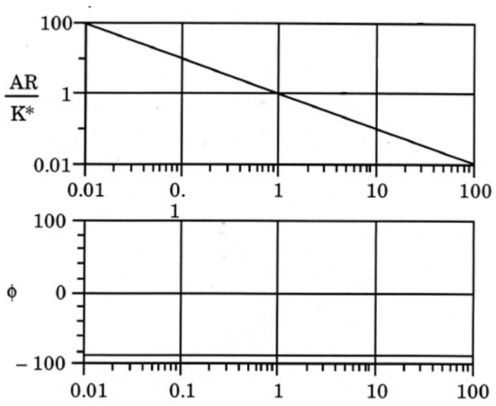


- **Integrating process**

$$G(s) = \frac{1}{As} \qquad G(j\omega) = \frac{1}{jA\omega} = -\frac{1}{A\omega}j$$

$$AR_N = |G(j\omega)| = \frac{1}{A\omega}$$

$$\phi = \measuredangle G(j\omega) = \tan^{-1}(-\frac{1}{0 \cdot \omega}) = -\frac{\pi}{2}$$

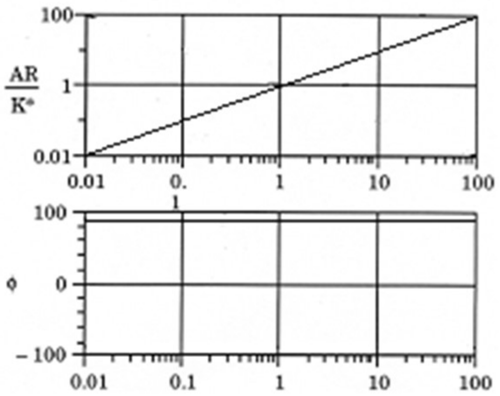


- **Differentiator**

$$G(s) = As \qquad G(j\omega) = jA\omega$$

$$AR_N = |G(j\omega)| = A\omega$$

$$\phi = \measuredangle G(j\omega) = \tan^{-1}(\frac{1}{0 \cdot \omega}) = \frac{\pi}{2}$$



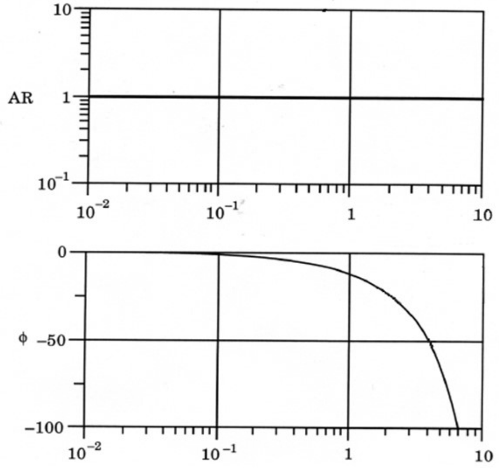
- **Pure delay process**

$$G(s) = e^{-\theta s}$$

$$G(j\omega) = e^{-j\theta\omega} = \cos\theta\,\omega - j\sin\theta\,\omega$$

$$AR = |G(j\omega)| = 1$$

$$\phi = \measuredangle G(j\omega) = -\tan^{-1}\tan\theta\,\omega = -\theta\omega$$



SKETCHING BODE PLOT

$$G(s) = \frac{G_a(s)G_b(s)G_c(s)\cdots}{G_1(s)G_2(s)G_3(s)\cdots} \qquad G(j\omega) = \frac{G_a(j\omega)G_b(j\omega)G_c(j\omega)\cdots}{G_1(j\omega)G_2(j\omega)G_3(j\omega)\cdots}$$

$$|G(j\omega)| = \frac{|G_a(j\omega)||G_b(j\omega)||G_c(j\omega)|\cdots}{|G_1(j\omega)||G_2(j\omega)||G_3(j\omega)|\cdots}$$

$$\begin{aligned}\measuredangle G(j\omega) = {} & \measuredangle G_a(j\omega) + \measuredangle G_b(j\omega) + \measuredangle G_c(j\omega) + \cdots \\ & -\measuredangle G_1(j\omega) - \measuredangle G_2(j\omega) - \measuredangle G_3(j\omega) - \cdots\end{aligned}$$

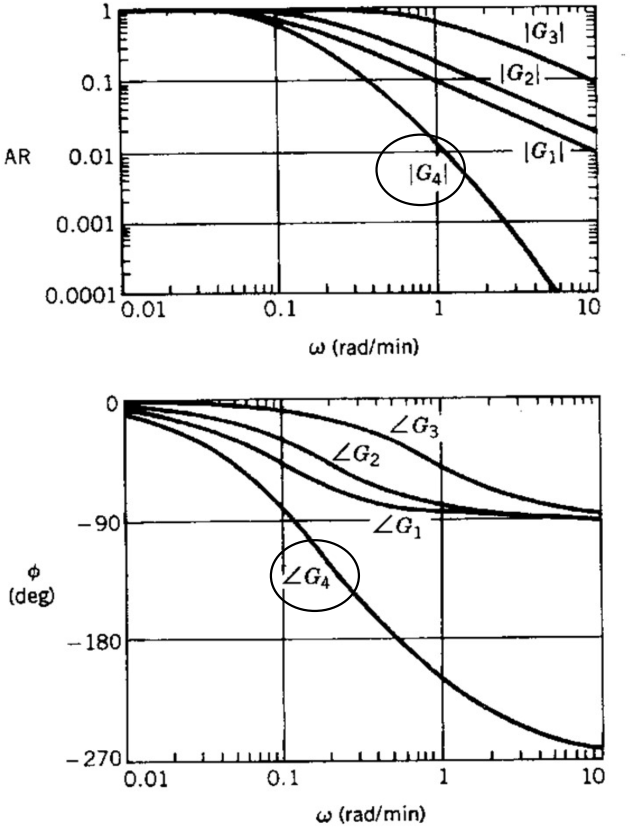
- **Bode diagram**
  - **AR vs. frequency in log-log plot**
  - **PA vs. frequency in semi-log plot**
  - **Useful for**
    - **Analysis of the response characteristics**
    - **Stability of the closed-loop system only for open-loop stable systems with phase angle curves exhibit a single critical frequency.**

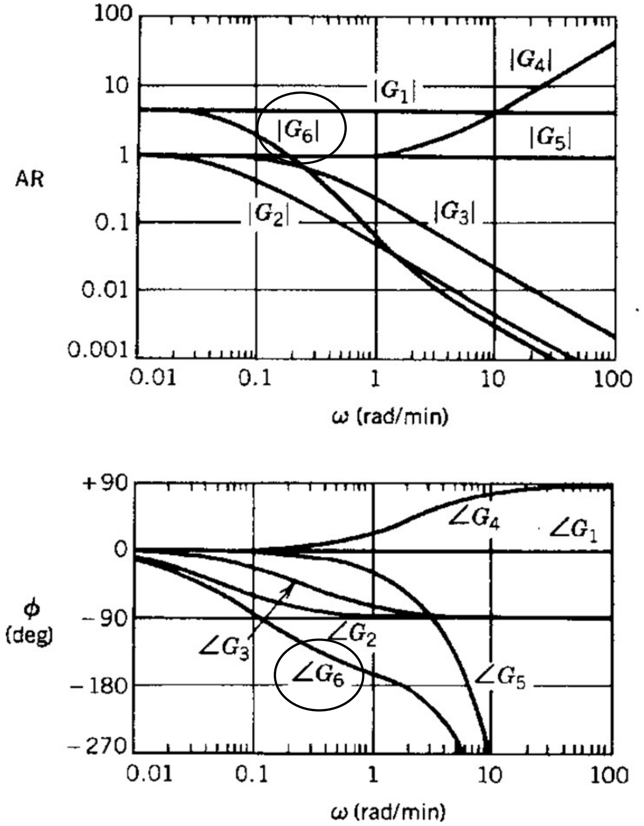
- **Amplitude Ratio on log-log plot**
  - Start from steady-state gain at $\omega = 0$. If G_{OL} includes either integrator or differentiator it starts at ∞ or 0.
  - Each first-order lag (lead) adds to the slope –1 (+1) starting at the corner frequency.
  - Each integrator (differentiator) adds to the slope –1 (+1) starting at zero frequency.
  - A delays does not contribute to the AR plot.
- **Phase angle on semi-log plot**
  - Start from 0° or -180° at $\omega = 0$ depending on the sign of steady-state gain.
  - Each first-order lag (lead) adds 0° to phase angle at $\omega = 0$, adds -90° (+90°) to phase angle at $\omega = \infty$, and adds -45° (+45°) to phase angle at corner frequency.
  - Each integrator (differentiator) adds -90° (+90°) to the phase angle for all frequency.
  - A delay adds $-\theta\omega$ to phase angle depending on the frequency.

Examples

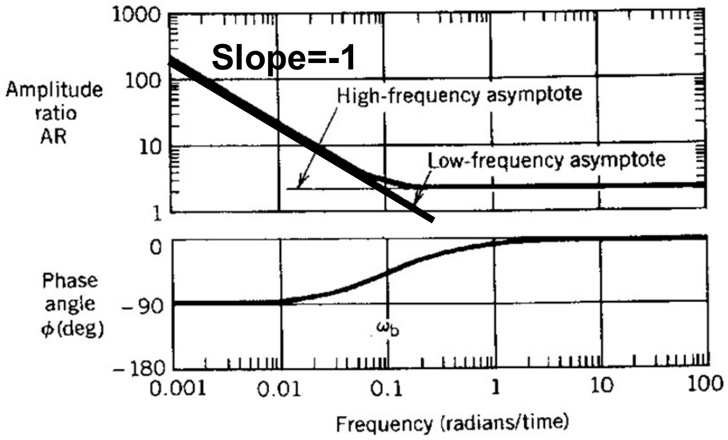
1. $$G(s)=\frac{K}{\underbrace{(10s+1)}_{G_1}\underbrace{(5s+1)}_{G_2}\underbrace{(s+1)}_{G_3}}$$

2. $$G(s)=\frac{\overbrace{5}^{G_1}\overbrace{(0.5s+1)}^{G_4}\overbrace{e^{-0.5s}}^{G_5}}{\underbrace{(20s+1)}_{G_2}\underbrace{(4s+1)}_{G_3}}$$





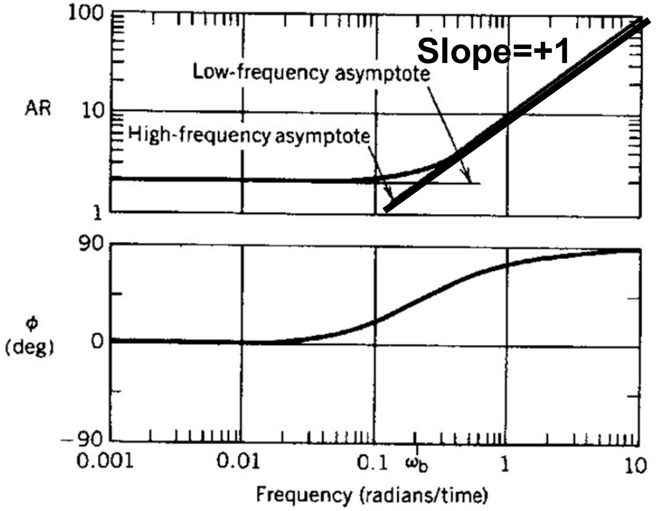
3. PI: $G(s) = K_C\left(1 + \frac{1}{\tau_I s}\right)$



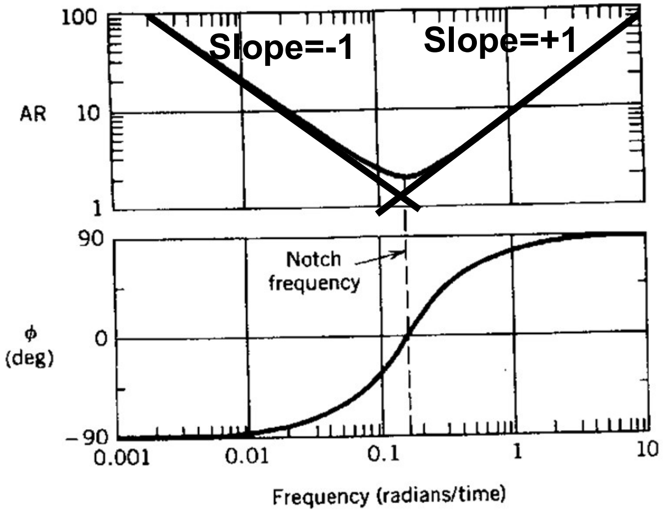
$\omega_b = 1/\tau_I$ at $\phi = -45°$

4. PD: $G(s) = K_C(1 + \tau_D s)$

$\omega_b = 1/\tau_D$ at $\phi = 45°$



5. PID: $G(s) = K_C\left(1 + \frac{1}{\tau_I s} + \tau_D s\right)$

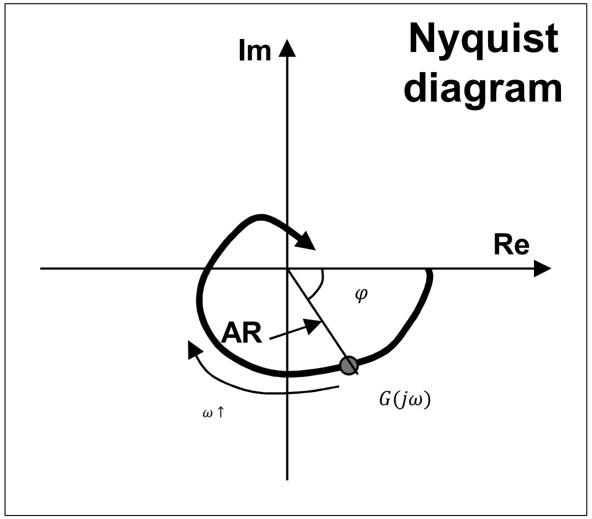


$\omega_{Notch} = \frac{1}{\sqrt{\tau_I \tau_D}}$ **at** $\phi = 0°$

NYQUIST DIAGRAM

- **Alternative representation of frequency response**
- **Polar plot of $G(j\omega)$ (ω is implicit)**

$$G(j\omega) = \mathrm{Re}[\,G(j\omega)] + j\,\mathrm{Im}[\,G(j\omega)]$$



- **Compact (one plot)**
- **Wider applicability of stability analysis than Bode plot**
- **High frequency characteristics will be shrunk near the origin.**
  - **Inverse Nyquist diagram: polar plot of $1/G(j\omega)$**
- **Combination of different transfer function components is not easy as with Nyquist diagram as with Bode plot.**