CBE507 LECTURE III
Controller Design Using State-space Methods

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Overview

- **States**
  - What characterize a system.
  - The internal state variables are the smallest possible subset of system variables that can represent the entire state of the system at any given time.
  - State variables must be linearly independent.
- **State-space model**

$$\dot{\mathbf{x}} = d\mathbf{x}/dt = \mathbf{f}(\mathbf{x}, \mathbf{u}, \mathbf{d})$$

$$\text{where } \mathbf{x} = [x_1, \cdots, x_n]^T, \mathbf{u} = [u_1, \cdots, u_m]^T, \mathbf{d} = [d_1, \cdots, d_l]^T$$

  - x: states
  - u: inputs
  - d: disturbances
  - y: outputs – can be a function of above, y=g(x,d,u)
  - If higher order derivatives exist, convert them to 1st order

Control Law Design

- **Linear state-space model (SISO)**
  - **Continuous-time version:** $\dot{\mathbf{x}} = \mathbf{F}\mathbf{x} + \mathbf{G}u$

    $y = \mathbf{H}\mathbf{x} + \mathbf{J}u$
  - **Discrete-time version:**

    $\mathbf{x}(k+1) = \mathbf{\Phi}\mathbf{x}(k) + \mathbf{\Gamma}u(k)$

    $y(k) = \mathbf{H}\mathbf{x}(k) + \mathbf{J}u(k)$
  - **where**

$$\mathbf{\Phi} = e^{\mathbf{F}\Delta t}, \quad \mathbf{\Gamma} = \int_0^{\Delta t} e^{\mathbf{F}\eta} d\eta \mathbf{G}$$

$$e^{-\mathbf{F}t}\dot{\mathbf{x}} - e^{-\mathbf{F}t}\mathbf{F}\mathbf{x} = e^{-\mathbf{F}t}\mathbf{G}u \Rightarrow \int_t^{t+\Delta t} d(e^{-\mathbf{F}\tau}\mathbf{x}) = \int_t^{t+\Delta t} e^{-\mathbf{F}\tau}\mathbf{G}u d\tau$$

$$e^{-\mathbf{F}t-\mathbf{F}\Delta t}\mathbf{x}(t+\Delta t) - e^{-\mathbf{F}t}\mathbf{x}(t) = \int_t^{t+\Delta t} e^{-\mathbf{F}\tau} u d\tau \mathbf{G}$$

$$\mathbf{x}(t+\Delta t) = e^{\mathbf{F}\Delta t}\mathbf{x}(t) + \int_t^{t+\Delta t} e^{-\mathbf{F}(\tau - t - \Delta t)} u d\tau \mathbf{G} \quad (\eta = \tau - t - \Delta t)$$

- **Design steps**
  - **If all the states are available, use state feedback. (u=–Kx)**
  - **If a part of states are available, design a state estimator.**
  - **The state estimator is also called a state observer.**

Calculation of $e^{\mathbf{A}t}$

- **Taylor series expansion**

$$\mathbf{\Phi} = e^{\mathbf{F}t} = \mathbf{I} + \mathbf{F}t + \mathbf{F}^2 t^2/2! + \mathbf{F}^3 t^3/3! + \cdots = \mathbf{I} + \mathbf{F}t\mathbf{\Psi}$$

$$\text{where } \mathbf{\Psi} = \mathbf{I} + \mathbf{F}t/2! + \mathbf{F}^2 t^2/3! + \mathbf{F}^3 t^3/4! + \cdots$$

$$\approx \mathbf{I} + \mathbf{F}t/2(\mathbf{I} + \mathbf{F}t/3(\cdots \mathbf{F}t/(N-1)(\mathbf{I} + \mathbf{F}t/N))\cdots)$$

- **Calculation of Φ and Γ**

$$\mathbf{\Phi} = e^{\mathbf{F}\Delta t} = \mathbf{I} + \mathbf{F}\Delta t\mathbf{\Psi}$$

$$\mathbf{\Gamma} = \int_0^{\Delta t} e^{\mathbf{F}\eta} d\eta \mathbf{G} \approx \sum_{k=0}^{\infty} \frac{\mathbf{F}^k \Delta t^{k+1}}{(k+1)!}\mathbf{G} = \sum_{k=0}^{\infty} \frac{\mathbf{F}^k \Delta t^k}{(k+1)!}\Delta t\mathbf{G} = \mathbf{\Psi}\Delta t\mathbf{G}$$

- **Example (Double integrator)**

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \quad y = \begin{bmatrix} 1 & 0 \end{bmatrix}\mathbf{x} \qquad (\mathbf{F}^2 = 0)$$

$$\mathbf{\Phi} = e^{\mathbf{F}\Delta t} = \mathbf{I} + \mathbf{F}\Delta t + \mathbf{F}^2 \Delta t^2/2! + \cdots = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\Delta t = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{\Gamma} = (\mathbf{I} + \mathbf{F}\Delta t/2)\Delta t\mathbf{G} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\Delta t + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\frac{\Delta t^2}{2} \right)\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \Delta t^2/2 \\ \Delta t \end{bmatrix}$$

- **State feedback control**

$$\mathbf{x}(k+1)=\mathbf{\Phi x}(k)-\mathbf{\Gamma K x}(k)\Rightarrow(z\mathbf{I}-\mathbf{\Phi}+\mathbf{\Gamma K})\mathbf{X}(z)=0$$

  - **Characteristic equation:** $\left|z\mathbf{I}-\mathbf{\Phi}+\mathbf{\Gamma K}\right|=0$

- **Pole placement example (Double integrator)**
  - **Desired pole location:** z=0.8 ± j0.25 **with** Δt=0.1

$$\left|z\mathbf{I}-\mathbf{\Phi}+\mathbf{\Gamma K}\right|=\left|z\begin{bmatrix}1&0\\0&1\end{bmatrix}-\begin{bmatrix}1&\Delta t\\0&1\end{bmatrix}+\begin{bmatrix}\Delta t^2/2\\\Delta t\end{bmatrix}\begin{bmatrix}K_1&K_2\end{bmatrix}\right|=0$$

$$z^2+(\Delta tK_2+\Delta t^2K_1/2-2)z+\Delta t^2K_1/2-\Delta tK_2+1=0$$

$$\text{Desired char. eqn: } z^2-1.6z+0.7=0$$

$$\Delta tK_2+\Delta t^2K_1/2-2=-1.6$$

$$z+\Delta t^2K_1/2-\Delta tK_2+1=0.7 \qquad \Rightarrow K_1=10, K_2=3.5$$

Control Canonical Form

- **Characteristic polynomial**
  - $\alpha(z)=z^n+a_1z^{n-1}+\ldots a_{n-1}z+a_n$
- **Control Canonical form**

$$\mathbf{x}(k+1)=\mathbf{\Phi}_c\mathbf{x}(k)+\mathbf{\Gamma}_c u(k)$$
$$y(k)=\mathbf{H}_c\mathbf{x}(k)$$

$$\mathbf{\Phi}_c=\begin{bmatrix} -a_1 & -a_2 & & & -a_n \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \cdots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix} \quad \mathbf{\Gamma}_c=\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \mathbf{H}_c=\begin{bmatrix} b_1 & b_2 & b_3 & \cdots & b_n \end{bmatrix}$$

- **State-feedback control**

$$\mathbf{x}(k+1)=\mathbf{\Phi}_c\mathbf{x}(k)-\mathbf{\Gamma}_c\mathbf{K}\mathbf{x}(k)=(\mathbf{\Phi}_c-\mathbf{\Gamma}_c\mathbf{K})\mathbf{x}(k)$$

$$\mathbf{\Phi}_c-\mathbf{\Gamma}_c\mathbf{K}=\begin{bmatrix} -a_1-K_1 & -a_2-K_2 & & \cdots & -a_n-K_n \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \cdots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix} \Rightarrow \alpha(z)=z^n+(a_1+K_1)z^2+\cdots+(a_n+K_n)$$

- **How to convert to canonical form**
  - **It is possible when the system is controllable.**
  - **Controllability**
    - **The system (Φ, Γ) is controllable if for every n^{th}-order polynomial $\alpha_c(z)$, there exists a control low u=−Kx such that the characteristic polynomial of ($\Phi-\Gamma$K) is $\alpha_c(z)$.**
      - **The pair (Φ, Γ) is controllable if and only if the rank of $C=[\Gamma\ \Phi\Gamma\ \Phi^2\Gamma\ \ldots\ \Phi^{n-1}\Gamma]$ is n.**
    - **The system (Φ, Γ) is controllable if for every x_0 and x_1 there is finite N and a sequence of control u_0, u_1, …, u_N such that if the system has state x_0 at k=0, it is forced to state x_1 at k=N.**
    - **The system (Φ, Γ) is controllable if every mode in Φ is connected to the control input.**

Controllability Matrix

- **Controllability matrix,** C=[Γ ΦΓ $\Phi^2\Gamma$ … $\Phi^{N\text{-}1}\Gamma$] **should be invertible for system to be controllable.**

$$\mathbf{x}(1) = \mathbf{\Phi}\mathbf{x}(0) + \mathbf{\Gamma}u(0)$$

$$\mathbf{x}(2) = \mathbf{\Phi}\mathbf{x}(1) + \mathbf{\Gamma}u(1) = \mathbf{\Phi}^2\mathbf{x}(0) + \mathbf{\Phi}\mathbf{\Gamma}u(0) + \mathbf{\Gamma}u(1)$$

$$\vdots$$

$$\mathbf{x}(N) = \mathbf{\Phi}^N\mathbf{x}(0) + \sum_{j=0}^{N-1}\mathbf{\Phi}^{N-1-j}\mathbf{\Gamma}u(j)$$

$$\Rightarrow \begin{bmatrix} \mathbf{\Gamma} & \mathbf{\Phi}\mathbf{\Gamma} & \cdots & \mathbf{\Phi}^{N-1}\mathbf{\Gamma} \end{bmatrix} \begin{bmatrix} u(N-1) \\ u(N-2) \\ \vdots \\ u(0) \end{bmatrix} = \mathbf{x}(N) - \mathbf{\Phi}^N\mathbf{x}(0)$$

- **Ackermann's formula**
  - **Satisfactory for SISO systems of order less than 10 and can handle systems with repeated roots.**

$$\mathbf{K} = \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{\Gamma} & \mathbf{\Phi}\mathbf{\Gamma} & \cdots & \mathbf{\Phi}^{N-1}\mathbf{\Gamma} \end{bmatrix}^{-1} \alpha_c(\mathbf{\Phi})$$

  - **Selecting <u>the desired characteristic polynomial</u>, $\alpha_c(z)$ should be based on system evaluations such as step response and etc.**

Estimator Design

- **If the full states are not available, unmeasured states should be estimated based on the output measurements to use state feedback strategy.**
- **Model of plant dynamics**

$$\bar{\mathbf{x}}(k+1) = \mathbf{\Phi}\bar{\mathbf{x}}(k) + \mathbf{\Gamma}u(k)$$

  - **The model (Φ, Γ) and $u(k)$ are known and the estimate of the initial condition x(0) is assumed to be obtained.**
  - **The error dynamics:** $\tilde{\mathbf{x}}(k+1) = \mathbf{\Phi}\tilde{\mathbf{x}}(k) \quad (\tilde{\mathbf{x}} = \mathbf{x} - \bar{\mathbf{x}})$
  - **If Φ is unstable, the error will not converge to zero.**
- **Prediction estimator**

$$\bar{\mathbf{x}}(k+1) = \mathbf{\Phi}\bar{\mathbf{x}}(k) + \mathbf{\Gamma}u(k) + \mathbf{L}_p[y(k) - \mathbf{H}\bar{\mathbf{x}}(k)]$$

$$\Rightarrow \quad \tilde{\mathbf{x}}(k+1) = [\mathbf{\Phi} - \mathbf{L}_p\mathbf{H}]\tilde{\mathbf{x}}(k)$$

  - **By pole placement, the estimator gain, L_p can be designed to have desired error dynamics.**

- **Is it possible to find L_p for desirable error dynamics?**
  - **It is possible when the system is observable.**
  - **Observability**
    - **The system (Φ, H) is observable if for every n^{th}-order polynomial $\alpha_e(z)$, there exists a estimator gain L_p such that the characteristic equation of state error of the estimator is $\alpha_e(z)$.**
      - **The pair (Φ, H) is controllable if and only if the rank of $O=[H\ H\Phi\ H\Phi^2\ \dots\ H\Phi^{n-1}]$ is n.**
    - **The system (Φ, H) is observable if for any x_0, there is finite N such that x_0 can be computed from observation $y_0, y_1, \dots, y_{N-1}$.**
    - **The system (Φ, H) is observable if every dynamic mode in Φ is connected to the output y via H.**

Observability Matrix

- **Observability matrix,** O=[H; HΦ; HΦ$^{2;}$ ...; HΦ$^{N\text{-}1}$] **should be invertible for system to be observable.**

$$y(0) = \mathbf{Hx}(0)$$
$$y(1) = \mathbf{Hx}(1) = \mathbf{H\Phi x}(0)$$
$$\vdots$$
$$y(N-1) = \mathbf{H\Phi}^{N-1}\mathbf{x}(0)$$

$$\Rightarrow \begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(N-1) \end{bmatrix} = \begin{bmatrix} \mathbf{H} \\ \mathbf{H\Phi} \\ \vdots \\ \mathbf{H\Phi}^{N-1} \end{bmatrix} \mathbf{x}(0)$$

  - **The controllability and the observability are dual.**

- **Ackermann's formula**
  - **Satisfactory for SISO systems of order less than 10 and can handle systems with repeated roots.**

$$\mathbf{L}_p^T = \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{H}^T & \mathbf{\Phi}^T\mathbf{H}^T & \cdots & \left(\mathbf{\Phi}^T\right)^{(N-1)}\mathbf{H}^T \end{bmatrix}^{-1} \alpha_e(\mathbf{\Phi}^T)$$

  - **Selecting <u>the desired characteristic polynomial</u>, $\alpha_e(z)$ should be based on prediction error dynamics.**

Current Estimator

- **The previous form of state estimation of $\mathbf{x}(k)$ does not involve current measurement.**
- **Alternative estimator formulation**

$$\hat{\mathbf{x}}(k) = \mathbf{\Phi}\bar{\mathbf{x}}(k) + \mathbf{\Gamma}u(k) + \mathbf{L}_p[y(k) - \mathbf{H}\bar{\mathbf{x}}(k)]$$

$$\bar{\mathbf{x}}(k) = \mathbf{\Phi}\hat{\mathbf{x}}(k-1) + \mathbf{\Gamma}u(k-1)$$

  - **It is impossible to have measurement $y(k)$ and $u(k)$ calculation at the same time.**
  - **However, the calculation of $u(k)$ can be arranged to minimize computational delays by performing all calculations before the sample instant that do not directly depend on $y(k)$ measurement.**
  - **In this case, the observability matrix and the calculation of current estimator gain $\mathbf{L}_c$ should be modified accordingly.**

Reduced-Order Estimators

- **The direct measurements of states do not need to be estimated.**
  - **If there is significant noise in measurements, full state estimation can provide smoothing for the measured state and estimation of unmeasured states.**
  - **Division of states (*a*: directly measured)**

$$\begin{bmatrix} \mathbf{x}_a(k+1) \\ \mathbf{x}_b(k+1) \end{bmatrix} = \begin{bmatrix} \mathbf{\Phi}_{aa} & \mathbf{\Phi}_{ab} \\ \mathbf{\Phi}_{ba} & \mathbf{\Phi}_{bb} \end{bmatrix} \begin{bmatrix} \mathbf{x}_a(k) \\ \mathbf{x}_b(k) \end{bmatrix} + \begin{bmatrix} \mathbf{\Gamma}_a \\ \mathbf{\Gamma}_b \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}_a(k) \\ \mathbf{x}_b(k) \end{bmatrix}$$

Known input

$$\mathbf{x}_b(k+1) = \mathbf{\Phi}_{bb}\mathbf{x}_b(k) + \boxed{\mathbf{\Phi}_{ba}\mathbf{x}_a(k) + \mathbf{\Gamma}_b u(k)}$$

Known measurement

$$\boxed{\mathbf{x}_a(k+1) - \mathbf{\Phi}_{aa}\mathbf{x}_a(k) - \mathbf{\Gamma}_a u(k)} = \mathbf{\Phi}_{ab}\mathbf{x}_b(k)$$

$$\hat{\mathbf{x}}_b(k+1) = \mathbf{\Phi}_{bb}\hat{\mathbf{x}}_b(k) + \mathbf{\Phi}_{ba}\mathbf{x}_a(k) + \mathbf{\Gamma}_b u(k) + \mathbf{L}_r[\mathbf{x}_a(k+1) - \mathbf{\Phi}_{aa}\mathbf{x}_a(k) - \mathbf{\Gamma}_a u(k) - \mathbf{\Phi}_{ab}\hat{\mathbf{x}}_b(k)]$$

$$\tilde{\mathbf{x}}_b(k+1) = [\mathbf{\Phi}_{bb} - \mathbf{L}_r\mathbf{\Phi}_{ab}]\tilde{\mathbf{x}}_b(k)$$

- **Reduced-order estimator gain $\mathbf{L}_r$ can be selected so that the following characteristics equation have desired roots.**

$$\alpha_e(z) = \left| z\mathbf{I} - \mathbf{\Phi}_{bb} + \mathbf{L}_r \mathbf{\Phi}_{ab} \right| = 0$$

- **Or, use Ackermann's formula**

$$\mathbf{L}_r = \alpha_e(\mathbf{\Phi}_{bb}) \begin{bmatrix} \mathbf{\Phi}_{ab} \\ \mathbf{\Phi}_{ab}\mathbf{\Phi}_{bb} \\ \mathbf{\Phi}_{ab}\mathbf{\Phi}_{bb}^2 \\ \vdots \\ \mathbf{\Phi}_{ab}\mathbf{\Phi}_{bb}^{n-r} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Regulator Design: Combined control law and Estimator

- **The separation principle**
  - Feedback control based on the estimator

    $u(k) = -\mathbf{K}\bar{\mathbf{x}}(k)$

    $\mathbf{x}(k+1) = \mathbf{\Phi}\mathbf{x}(k) - \mathbf{\Gamma}\mathbf{K}\bar{\mathbf{x}}(k) = \mathbf{\Phi}\mathbf{x}(k) - \mathbf{\Gamma}\mathbf{K}(\mathbf{x}(k) - \tilde{\mathbf{x}}(k))$
  - Estimator

    $\tilde{\mathbf{x}}(k+1) = [\mathbf{\Phi} - \mathbf{L}_p\mathbf{H}]\tilde{\mathbf{x}}(k)$
  - Combination

    $$\begin{bmatrix} \tilde{\mathbf{x}}(k+1) \\ \mathbf{x}(k+1) \end{bmatrix} = \begin{bmatrix} \mathbf{\Phi} - \mathbf{L}_p\mathbf{H} & 0 \\ \mathbf{\Gamma}\mathbf{K} & \mathbf{\Phi} - \mathbf{\Gamma}\mathbf{K} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{x}}(k) \\ \mathbf{x}(k) \end{bmatrix}$$
  - Characteristic equation

    $\alpha_c(\mathrm{z})\alpha_e(\mathrm{z}) = \left|\mathrm{z}\mathbf{I} - \mathbf{\Phi} + \mathbf{\Gamma}\mathbf{K}\right|\left|\mathrm{z}\mathbf{I} - \mathbf{\Phi} + \mathbf{L}_p\mathbf{H}\right| = 0$
  - The characteristic poles of complete system consists of the combination of the estimator poles and the control poles
  - The estimator and the controller can be designed separately.

Use of Reference Input

- **So far, the state feedback was focus on driving the states to zero. (Regulator problem)**
- **How to incorporate set point**
  - **For a given set point r, find N_x that defines the desired value of states x_r. ($N_x r = x_r$)**

$$\mathbf{u} = -\mathbf{K}(\mathbf{x} - \mathbf{x}_r) + \mathbf{u}_{ss} = -\mathbf{K}(\mathbf{x} - \mathbf{N}_x\mathbf{r}) + \mathbf{N}_u\mathbf{r}$$

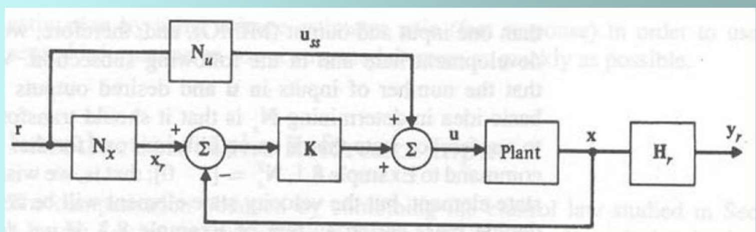
$$\text{where } \mathbf{u}_{ss} = \mathbf{N}_u\mathbf{r}$$

$$\mathbf{x}(k+1) = \mathbf{\Phi x}(k) + \mathbf{\Gamma u}(k) \Rightarrow \mathbf{x}_{ss} = \mathbf{\Phi x}_{ss} + \mathbf{\Gamma u}_{ss} \Rightarrow (\mathbf{\Phi} - \mathbf{I})\mathbf{x}_{ss} + \mathbf{\Gamma u}_{ss} = 0$$

  - **Some system output $y_r = H_r x$ will follow the desired reference.**

$$\mathbf{y}_r = \mathbf{H}_r\mathbf{x}_{ss} = \mathbf{H}_r\mathbf{N}_x\mathbf{r} = \mathbf{r} \Rightarrow \mathbf{H}_r\mathbf{N}_x = \mathbf{I}$$

$$\Rightarrow \begin{bmatrix} \mathbf{\Phi} - \mathbf{I} & \mathbf{\Gamma} \\ \mathbf{H}_r & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{N}_x \\ \mathbf{N}_u \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \end{bmatrix} \Rightarrow \begin{bmatrix} \mathbf{N}_x \\ \mathbf{N}_u \end{bmatrix} = \begin{bmatrix} \mathbf{\Phi} - \mathbf{I} & \mathbf{\Gamma} \\ \mathbf{H}_r & \mathbf{0} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \end{bmatrix}$$



Reference input with estimator

- **State-command structure**
  - Prediction estimator

$$\mathbf{u}(k) = -\mathbf{K}(\bar{\mathbf{x}}(k) - \mathbf{x}_r) + \mathbf{N}_u\mathbf{r} = -\mathbf{K}\bar{\mathbf{x}}(k) + \bar{\mathbf{N}}\mathbf{r}$$

where $\bar{\mathbf{N}} = \mathbf{N}_u + \mathbf{K}\mathbf{N}_x$

$$\begin{bmatrix} \mathbf{x}(k+1) \\ \tilde{\mathbf{x}}(k+1) \end{bmatrix} = \begin{bmatrix} \mathbf{\Phi} & -\mathbf{\Gamma K} \\ \mathbf{L}_p\mathbf{H} & \mathbf{\Phi} - \mathbf{\Gamma K} - \mathbf{L}_p\mathbf{H} \end{bmatrix} \begin{bmatrix} \mathbf{x}(k) \\ \tilde{\mathbf{x}}(k) \end{bmatrix} + \begin{bmatrix} \mathbf{\Gamma}\bar{\mathbf{N}} \\ \mathbf{\Gamma}\bar{\mathbf{N}} \end{bmatrix} r(k) + \begin{bmatrix} \mathbf{\Gamma}_1 \\ 0 \end{bmatrix} w(k)$$

  - Current estimator

$$\mathbf{u}(k) = -\mathbf{K}(\hat{\mathbf{x}}(k) - \mathbf{x}_r) + \mathbf{N}_u\mathbf{r} = -\mathbf{K}\hat{\mathbf{x}}(k) + \bar{\mathbf{N}}\mathbf{r}$$

$$\begin{bmatrix} \mathbf{x}(k+1) \\ \hat{\mathbf{x}}(k+1) \end{bmatrix} = \begin{bmatrix} \mathbf{\Phi} & -\mathbf{\Gamma K} \\ \mathbf{L}_c\mathbf{H}\mathbf{\Phi} & \mathbf{\Phi} - \mathbf{\Gamma K} - \mathbf{L}_c\mathbf{H}\mathbf{\Phi} \end{bmatrix} \begin{bmatrix} \mathbf{x}(k) \\ \hat{\mathbf{x}}(k) \end{bmatrix} + \begin{bmatrix} \mathbf{\Gamma}\bar{\mathbf{N}} \\ \mathbf{\Gamma}\bar{\mathbf{N}} + \mathbf{L}_c\mathbf{H}\mathbf{\Gamma}\bar{\mathbf{N}} \end{bmatrix} r(k) + \begin{bmatrix} \mathbf{\Gamma}_1 \\ \mathbf{L}_c\mathbf{H}\mathbf{\Gamma}_1 \end{bmatrix} w(k)$$

- **Output error command**

$$\bar{\mathbf{x}}(k+1) = (\mathbf{\Phi} - \mathbf{\Gamma K} - \mathbf{L}_p\mathbf{H})\bar{\mathbf{x}}(k) + \mathbf{L}_p(y(k) - r(k))$$

$$\begin{bmatrix} \mathbf{x}(k+1) \\ \bar{\mathbf{x}}(k+1) \end{bmatrix} = \begin{bmatrix} \mathbf{\Phi} & -\mathbf{\Gamma K} \\ \mathbf{L}_p\mathbf{H} & \mathbf{\Phi} - \mathbf{\Gamma K} - \mathbf{L}_p\mathbf{H} \end{bmatrix} \begin{bmatrix} \mathbf{x}(k) \\ \bar{\mathbf{x}}(k) \end{bmatrix} + \begin{bmatrix} 0 \\ -\mathbf{L}_p \end{bmatrix} r(k) + \begin{bmatrix} \mathbf{\Gamma}_1 \\ 0 \end{bmatrix} w(k)$$

Integral Control

- **Integral control by state augmentation**
  - **Augment the state with x_I, the integral error, $e=y-r$.**

$$\mathbf{x}(k+1)=\mathbf{\Phi x}(k)+\mathbf{\Gamma}u(k)+\mathbf{\Gamma}_1 w(k)$$

$$y(k)=\mathbf{Hx}(k)$$

$$x_I(k+1)=x_I(k)+e(k)=x_I(k)+\mathbf{Hx}(k)-r(k)$$

$$\Rightarrow\begin{bmatrix} x_I(k+1) \\ \mathbf{x}(k+1)\end{bmatrix}=\begin{bmatrix} 1 & \mathbf{H} \\ 0 & \mathbf{\Phi}\end{bmatrix}\begin{bmatrix} x_I(k) \\ \mathbf{x}(k)\end{bmatrix}+\begin{bmatrix} 0 \\ \mathbf{\Gamma}\end{bmatrix}u(k)-\begin{bmatrix} 1 \\ \mathbf{0}\end{bmatrix}r(k)$$

  - **Control law**

$$\mathbf{u}(k)=-\begin{bmatrix} K_I & \mathbf{K}\end{bmatrix}\begin{bmatrix} x_I(k) \\ \mathbf{x}(k)\end{bmatrix}+\mathbf{KN}_x r(k)$$

  - **The integral is replacing the feedforward term, $\mathbf{N}_x$. Also, it has the additional role of eliminating errors due to w.**

Disturbance Estimation

- **Disturbance rejection**
  - An alternative approach to state augmentation is to estimate the disturbance signal in the estimator and then to use that estimate in the control law so as to force the error to zero.
  - This approach yields results that are equivalent to integral control when the disturbance is constant.
- **Disturbance modeling**
  - Disturbance other than constant biases can be modeled.

$$\mathbf{x}_d(k+1) = \mathbf{\Phi}_d \mathbf{x}_d(k)$$

$$w(k) = \mathbf{H}_d \mathbf{x}_d(k)$$

$$\begin{bmatrix} \mathbf{x}(k+1) \\ \mathbf{x}_d(k+1) \end{bmatrix} = \begin{bmatrix} \mathbf{\Phi} & -\mathbf{\Gamma}_1 \mathbf{H}_d \\ \mathbf{0} & \mathbf{\Phi}_d \end{bmatrix} \begin{bmatrix} \mathbf{x}(k) \\ \mathbf{x}_d(k) \end{bmatrix} + \begin{bmatrix} \mathbf{\Gamma} \\ \mathbf{0} \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} \mathbf{H} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x}(k) \\ \mathbf{x}_d(k) \end{bmatrix}$$

- **All the ideas of state estimation still apply to reconstruct the state consisting x and x_d, provided the system is observable.**
- **However, the control gain matrix K is not obtained using the augmented model.**
- **The augmented system will always be uncontrollable since the disturbance is not influenced by control input by no means.**
- **The estimation of *w* will be used in a feedforward control scheme to eliminate its effect on steady-state errors.**
- **This basic idea works if *w* is a constant, a sinusoid, or any combination of functions that can be generated by a linear model.**
- **The only constraint is that the disturbance state x_d be observable.**

Effect of Delays

- **Sensor delay**
  - System: $\mathbf{x}(k+1) = \mathbf{\Phi}\mathbf{x}(k) + \mathbf{\Gamma}u(k)$

    $y(k) = \mathbf{H}\mathbf{x}(k)$
  - Delayed output: $y_{1d}(k+1) = y(k)$

    $y_{2d}(k+1) = y_{1d}(k)$

$$\Rightarrow \begin{bmatrix} \mathbf{x}(k+1) \\ y_{1d}(k+1) \\ y_{2d}(k+1) \end{bmatrix} = \begin{bmatrix} \mathbf{\Phi} & \mathbf{0} & \mathbf{0} \\ \mathbf{H} & 0 & 0 \\ \mathbf{0} & 1 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}(k) \\ y_{1d}(k) \\ y_{2d}(k) \end{bmatrix} + \begin{bmatrix} \mathbf{\Gamma} \\ 0 \\ 0 \end{bmatrix} u(k)$$

$$y_d(k) = \begin{bmatrix} \mathbf{0} & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{x}(k) \\ y_{1d}(k) \\ y_{2d}(k) \end{bmatrix}$$

  - **The augmented system matrix will cause problem in calculating gains for current estimator.**

- **Actuator delay**

$$\begin{bmatrix} \mathbf{x}(k+1) \\ u_d(k+1) \end{bmatrix} = \begin{bmatrix} \mathbf{\Phi} & \mathbf{\Gamma} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}(k) \\ u_d(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k) + \begin{bmatrix} \mathbf{\Gamma}_1 \\ 0 \end{bmatrix} w(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}(k) \\ u_d(k) \end{bmatrix}$$