CBE495 LECTURE III
CONTROL OF MULTI INPUT MULTI OUTPUT PROCESSES

Professor Dae Ryook Yang

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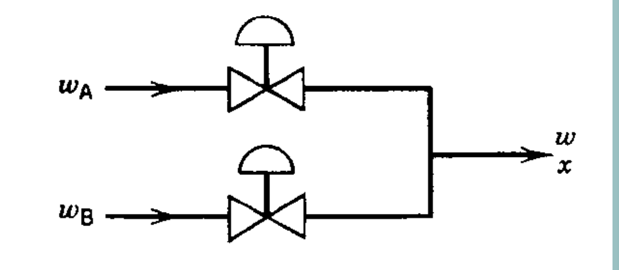
Dept. of Chemical and Biological Engineering
Korea University

Multi-Input Multi-output (MIMO) Processes

- **Single-input single-output (SISO) processes**
  - One CV and one MV: No need of pairing
- **Multi-input Multi-output (MIMO) processes**
  - Several CV's and several MV's
  - SIMO and MISO
  - The numbers of CV's and MV's are not necessary same.
  - One MV affects all or some of CV's. (process interaction)
  - *Pairing*: Which MV will control which CV?
  - *Control loop interaction*: One control loop affects the other control loops.
  - *Multiloop control*: Multiple SISO controllers are applied.
  - *Multivariable control*: All MV's will be manipulated to all or some CV's.

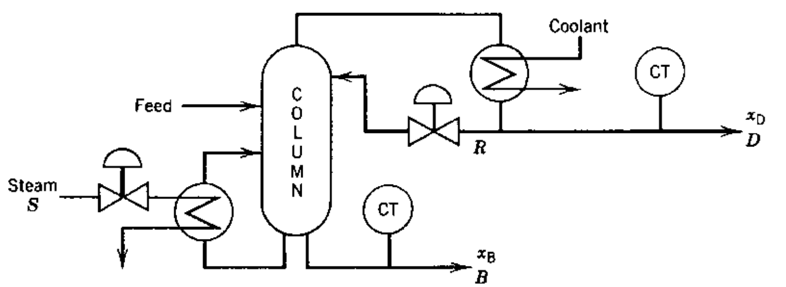
MIMO Process Examples

- **Inline blending system**
  - Component flows affect both product flow and composition.



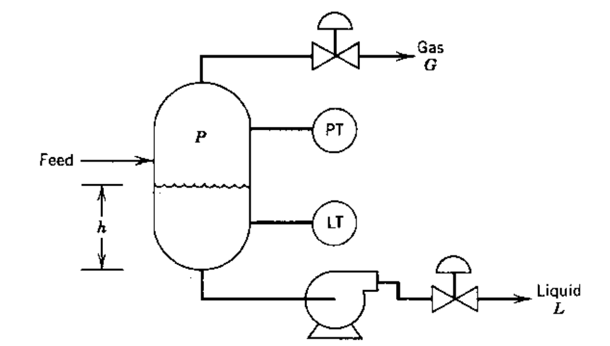
(a) In-line blending system

- **Distillation column**
  - Steam and reflux flows affect both top and bottom product compositions.



(b) Distillation column

- **Gas-liquid separator**
  - Gas and liquid product flows affect both tank level and pressure.



(c) Gas–liquid separator

Control loop interaction

- **2x2 control problem**
  - Two-input and two-output process
  - Transfer function (superposition principle for linear process)

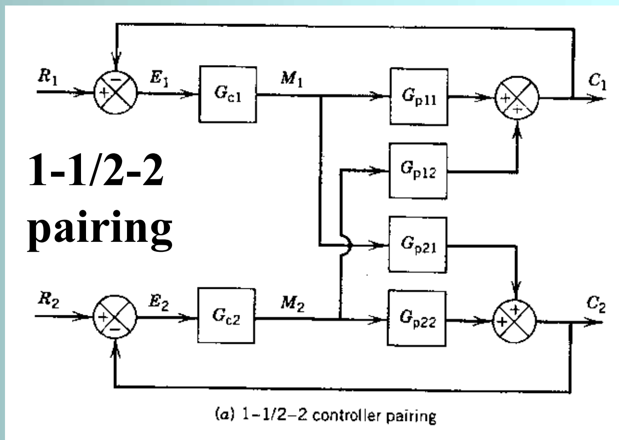
$$y_1(s) = G_{p_{11}}(s)u_1(s) + G_{p_{12}}(s)u_2(s)$$
$$y_2(s) = G_{p_{21}}(s)u_1(s) + G_{p_{22}}(s)u_2(s)$$

$$\begin{bmatrix} y_1(s) \\ y_2(s) \end{bmatrix} = \begin{bmatrix} G_{p_{11}}(s) & G_{p_{12}}(s) \\ G_{p_{21}}(s) & G_{p_{22}}(s) \end{bmatrix} \begin{bmatrix} u_1(s) \\ u_2(s) \end{bmatrix}$$

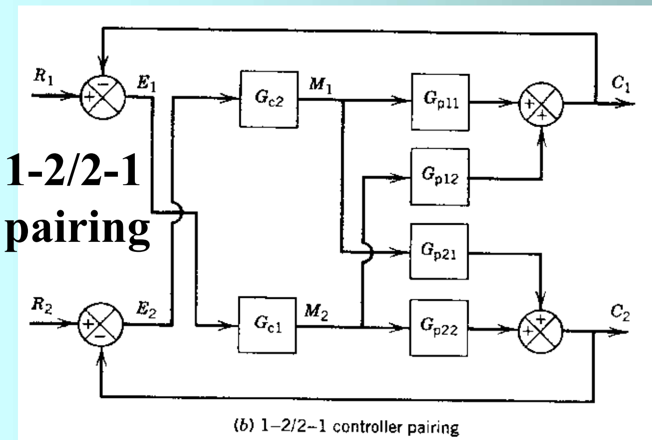
Process interactions

$$\left(\mathbf{y}(s) = \mathbf{G_p}(s)\mathbf{u}(s)\right)$$

  - Multiloop control schemes for 2x2 process



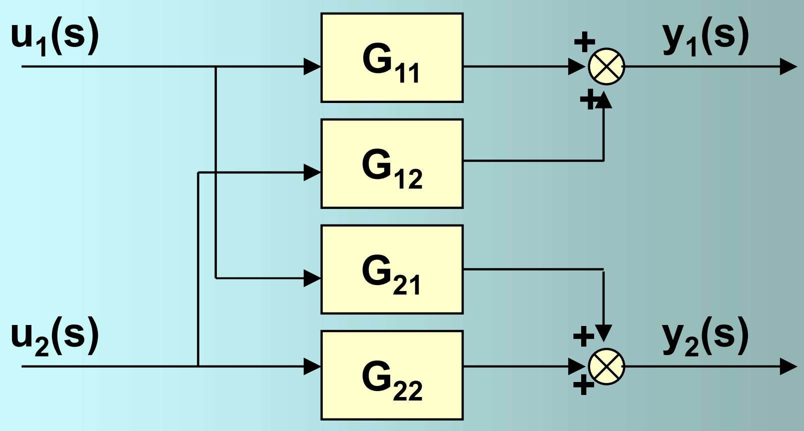
(a) 1-1/2-2 controller pairing



(b) 1-2/2-1 controller pairing

- **Open-loop transfer function for G_{p22}**

$$\frac{y_2(s)}{u_2(s)} = G_{p22}$$



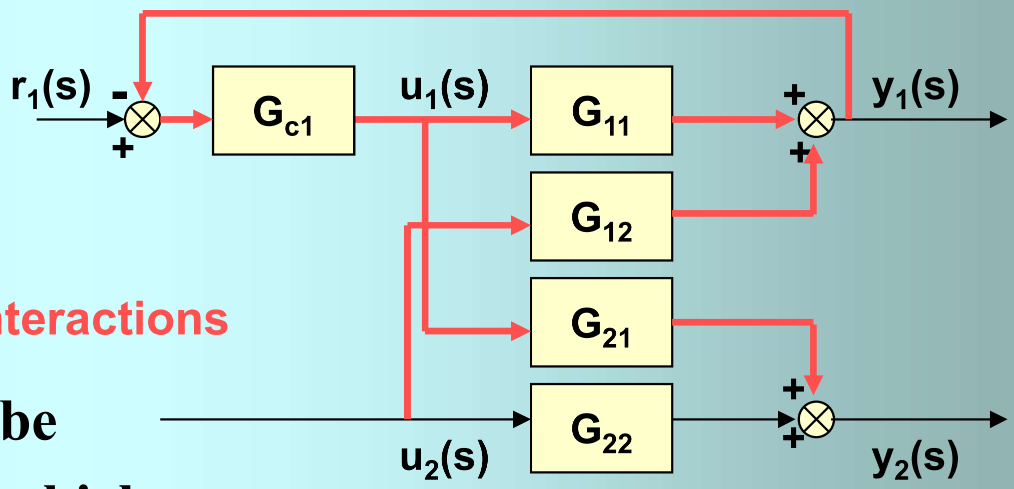
- **When y_1-u_1 loop is closed. (automatic mode)**

$$y_2 = G_{p22}u_2 + G_{p21}u_1$$

$$= G_{p22}u_2 - \frac{G_{p21}G_{p12}G_{c1}}{1+G_{c1}G_{p11}}u_2$$

$$\frac{y_2}{u_2} = G_{p22} - \frac{G_{p21}G_{p12}G_{c1}}{1+G_{c1}G_{p11}}$$

Control loop interactions

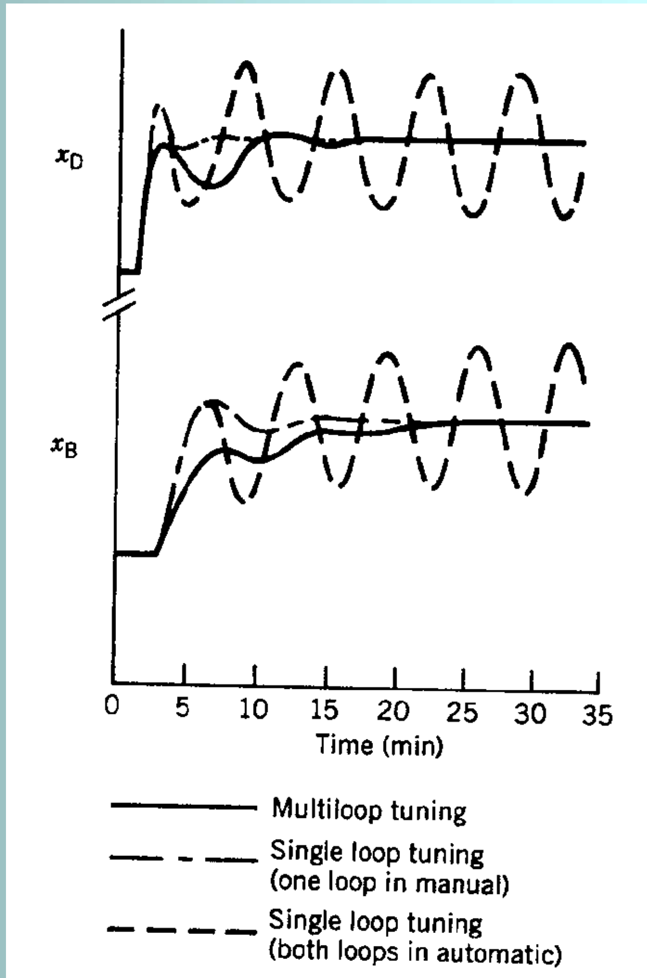


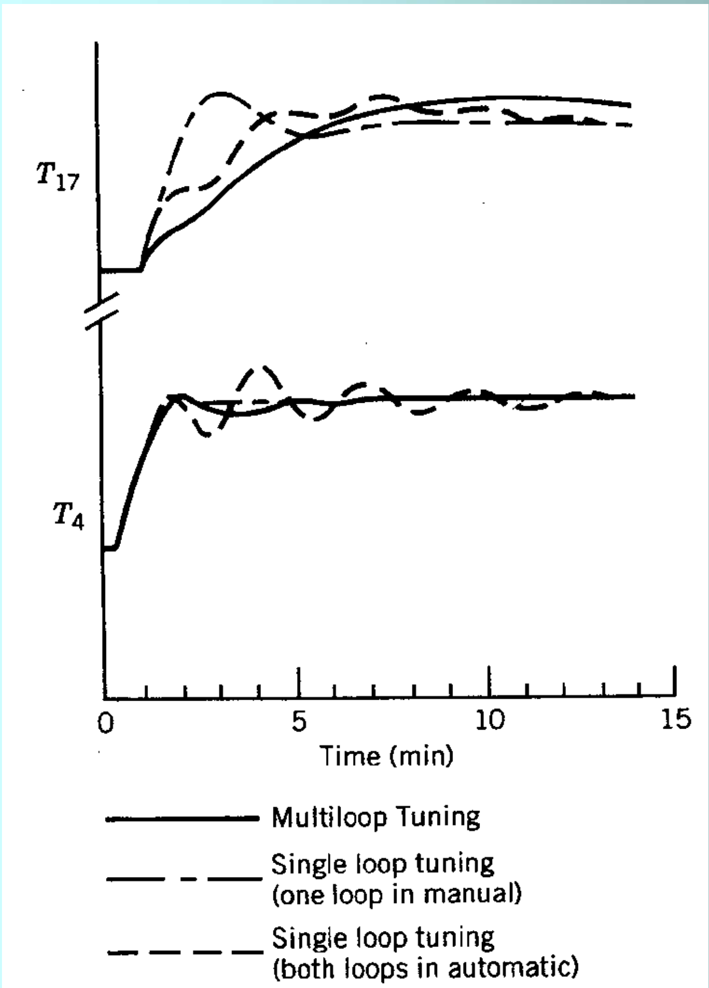
– **The controller G_{c2} should be designed based on the TF which is altered by the other control loop.**

- **Examples**

$$\begin{bmatrix} X_D \\ X'_B \end{bmatrix} = \begin{bmatrix} \dfrac{12.8e^{-s}}{16.7s+1} & \dfrac{-18.9e^{-3s}}{21s+1} \\ \dfrac{6.6e^{-7s}}{10.9s+1} & \dfrac{-19.4e^{-3s}}{14.4s+1} \end{bmatrix} \begin{bmatrix} R \\ S \end{bmatrix}$$

$$\begin{bmatrix} T_{17} \\ T_4 \end{bmatrix} = \begin{bmatrix} \dfrac{-2.16e^{-s}}{8.5s+1} & \dfrac{1.26e^{-0.3s}}{7.05s+1} \\ \dfrac{-2.75e^{-1.8s}}{8.25s+1} & \dfrac{4.28e^{-0.35s}}{9.0s+1} \end{bmatrix} \begin{bmatrix} R \\ S \end{bmatrix}$$





2x2 Multiloop Control

- **Closed-loop TF**

$$\frac{y_2}{r_2} = \frac{G_{c2}\left(G_{p22} - \dfrac{G_{p21}G_{p12}G_{c1}}{1+G_{c1}G_{p11}}\right)}{1+G_{c2}\left(G_{p22} - \dfrac{G_{p21}G_{p12}G_{c1}}{1+G_{c1}G_{p11}}\right)} = \frac{G_{c2}G_{p22}(1+G_{c1}G_{p11}) - G_{c2}G_{p21}G_{p12}G_{c1}}{1+G_{c1}G_{p11}+G_{c2}G_{p22}(1+G_{c1}G_{p11}) - G_{c2}G_{p21}G_{p12}G_{c1}}$$

$$\frac{y_2}{r_2} = \frac{G_{c2}G_{p22} + G_{c1}G_{c2}(G_{p11}G_{p22} - G_{p21}G_{p12})}{(1+G_{c1}G_{p11})(1+G_{c2}G_{p22}) - G_{c1}G_{c2}G_{p21}G_{p12}}$$

$$\frac{y_1}{r_1} = \frac{G_{c1}G_{p11} + G_{c1}G_{c2}(G_{p11}G_{p22} - G_{p21}G_{p12})}{(1+G_{c1}G_{p11})(1+G_{c2}G_{p22}) - G_{c1}G_{c2}G_{p21}G_{p12}}$$

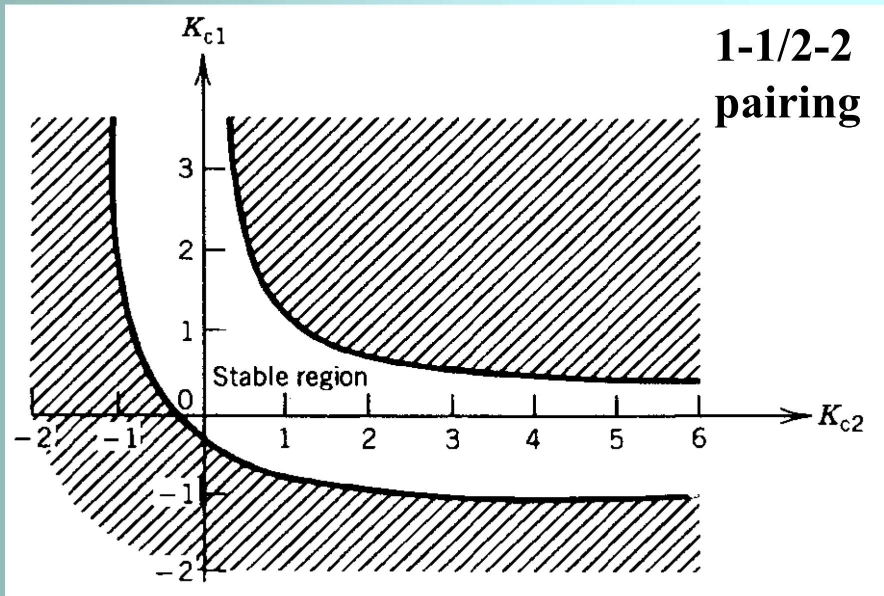
  - **Closed-loop stability depends on both G_{c1} and G_{c2}.**
  - **If either one or both of G_{p12} and G_{p21} are zero, the interaction term is vanished. The stability depends on two individual feedback control loops.**
  - **For example, if G_{p21} is zero, $G_{p12}\,U_2$ is a disturbance on y_1.**

- **Examples**

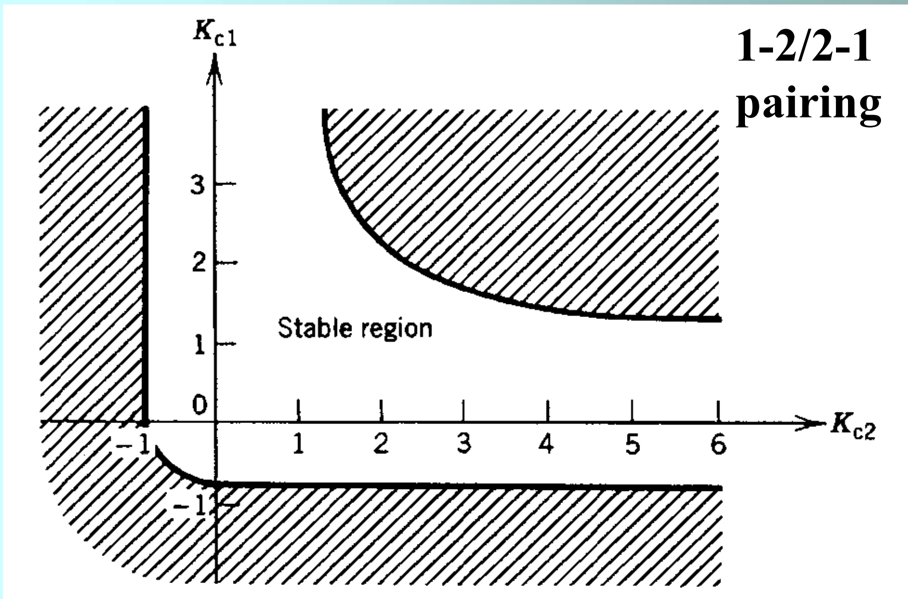
$$\mathbf{G_p} = \begin{bmatrix} \frac{2}{10s+1} & \frac{1.5}{s+1} \\ \frac{1.5}{s+1} & \frac{2}{10s+1} \end{bmatrix}$$

$$\mathbf{y} = \left[\mathbf{I} + \mathbf{G_p G_c}\right]^{-1} \mathbf{G_p G_c r} = \frac{Adj\left(\mathbf{I} + \mathbf{G_p G_c}\right)\mathbf{G_p G_c}}{\left|\mathbf{I} + \mathbf{G_p G_c}\right|}\mathbf{r}$$

- **Examine the poles from** $\left|\mathbf{I} + \mathbf{G_p G_c}\right| = 0$



$$\mathbf{G_c} = \begin{bmatrix} G_{c1} & 0 \\ 0 & G_{c2} \end{bmatrix}$$



$$\mathbf{G_c} = \begin{bmatrix} 0 & G_{c2} \\ G_{c1} & 0 \end{bmatrix}$$

Pairing of CV's and MV's

- **Bristol's relative gain array (RGA)**
  - ***Relative gain*** **is a measure of process interaction**

$$\lambda_{ij} = \frac{(\partial Y_i / \partial U_j)_U}{(\partial Y_i / \partial U_j)_Y} = \frac{\text{open-loop gain}}{\text{closed-loop gain}}$$

$$\Lambda = \begin{bmatrix} \lambda_{11} & \lambda_{12} & \cdots & \lambda_{1n} \\ \lambda_{21} & \lambda_{22} & & \lambda_{2n} \\ \vdots & & \ddots & \vdots \\ \lambda_{n1} & \lambda_{n2} & \cdots & \lambda_{nn} \end{bmatrix}$$

RGA

  - **The ratio between open-loop and closed-loop gains.**
  - **The** ***open-loop gain*****:** $(\partial y_i / \partial u_j)_u$ **is the gain between** y_i **and** u_j **while all loops are open.**
  - **The** ***closed-loop gain*****:** $(\partial y_i / \partial u_j)_y$ **is the gain between** y_i **and** u_j **while all other loops are closed.**
  - **Choose** λ_{ij} **so that it is close to unity or at least not negative for the multiloop pair.**

- **Properties of RGA**
  - **It is normalized since the sum of the elements in each row or column is one.** $\sum_{i=1}^{n} \lambda_{ij} = \sum_{j=1}^{n} \lambda_{ij} = 1$
  - **The relative gains are dimensionless and thus not affected by choice of units or scaling of variables.**
  - **The value of RG is a measure of steady-state interaction.**
  - $\lambda_{ij} = 1$ **implies that closed-loop gain is same as open-loop gain.**
  - $\lambda_{ij} = 0$ **implies that the *i*-th output is not affected by the *j*-th input in open-loop mode or closed-loop gain becomes infinity.**
  - **The value of** $1/\lambda_{ij}$ **represents the degree of alteration of open-loop gain when other loops are closed.**

$$K_P^{closed} = K_P^{open} / \lambda_{ij}$$

  - **The negative RG implies the closed-loop gain will be different in sign compared to open-loop gain. This pairing is potentially unstable and should be avoided.**

- **Proof of** $\sum_{j=1}^{n} \lambda_{ij} = 1$

$$\left(\delta Y_i\right)_Y = \sum_{j=1}^{n} \left(\frac{\partial y_i}{\partial u_j}\right)_u \left(\delta u_j\right)_y$$

$$\Rightarrow 1 = \sum_{j=1}^{n} \left(\frac{\partial y_i}{\partial u_j}\right)_u / \left(\frac{\delta y_i}{\delta u_j}\right)_y = \sum_{j=1}^{n} \lambda_{ij}$$

- **Proof of** $\sum_{i=1}^{n} \lambda_{ji} = 1$

$$\left(\delta U_j\right)_U = \sum_{i=1}^{n} \left(\frac{\partial u_j}{\partial y_i}\right)_y \left(\delta y_i\right)_u$$

$$\Rightarrow 1 = \sum_{i=1}^{n} \left(\frac{\partial y_i}{\partial u_j}\right)_u / \left(\frac{\delta y_i}{\delta u_j}\right)_y = \sum_{i=1}^{n} \lambda_{ij}$$

Calculating RGA

- 2x2 system
  - **All loops are open**

$$y_1 = K_{p11}u_1 + K_{p12}u_2$$
$$y_2 = K_{p21}u_1 + K_{p22}u_2$$

$$\left(\frac{\partial y_1}{\partial u_1}\right)_{u_2} = K_{P11}$$

  - **Second loop is closed (Y_2 is controlled perfectly controlled by U_2)**

$$y_2 = K_{p21}u_1 + K_{p22}u_2 = 0 \Rightarrow u_2 = -(K_{p21}/K_{p22})u_1$$

$$\therefore y_1 = K_{p11}u_1 - K_{p12}(K_{p21}/K_{p22})u_1$$

$$\left(\frac{\partial y_1}{\partial u_1}\right)_{y_2} = K_{P11} - \frac{K_{P12}K_{P21}}{K_{P22}}$$

$$\lambda_{11} = \left(\frac{\partial y_1}{\partial u_1}\right)_{u_2} / \left(\frac{\partial y_1}{\partial u_1}\right)_{y_2} = K_{P11} / \left(K_{P11} - \frac{K_{P12}K_{P21}}{K_{P22}}\right) = \frac{K_{P11}K_{P22}}{K_{P11}K_{P22} - K_{P12}K_{P21}}$$

$$\lambda_{11} = \frac{1}{1 - K_{P12}K_{P21}/K_{P11}K_{P22}}$$

$$\Lambda = \begin{bmatrix} \lambda_{11} & 1-\lambda_{11} \\ 1-\lambda_{11} & \lambda_{11} \end{bmatrix}$$

– **nxn system**

- **Except the *i*-th controller, all other control loops are closed.**

$$\begin{bmatrix} 0 \\ \vdots \\ y_i \\ \vdots \\ 0 \end{bmatrix} = \mathbf{K_p u} \quad \Rightarrow \quad \mathbf{u} = \mathbf{K_P}^{-1} \begin{bmatrix} 0 \\ \vdots \\ y_i \\ \vdots \\ 0 \end{bmatrix} \quad \Rightarrow \quad u_j = (\mathbf{K_P}^{-1})_{ji}\, y_i$$

$$\left(\frac{\partial y_i}{\partial u_j} \right)_Y = \frac{1}{(\mathbf{K_P}^{-1})_{ji}} = \frac{1}{(\mathbf{K_P}^{-1})_{ij}^T}$$

$$\therefore \lambda_{ij} = \left(\frac{\partial y_i}{\partial u_j} \right)_u \Big/ \left(\frac{\partial y_i}{\partial u_j} \right)_y = (\mathbf{K_P})_{ij} (\mathbf{K_P}^{-1})_{ji} = (\mathbf{K_P})_{ij} (\mathbf{K_P}^{-1})_{ij}^T$$

$$\Lambda = \mathbf{K_p} \otimes (\mathbf{K_p}^{-1})^T \quad (\otimes \text{ is an element-by-element multiplication})$$

$$\left(\Lambda \neq \mathbf{K_p} (\mathbf{K_p}^{-1})^T \right)$$

- **Implications of RGA elements**

1. $\lambda_{ij} = 1$ **: Indicating that the open-loop gain is identical to the closed-loop gain when Y_i and U_j are paired. This loop can be tuned independently. (Ideal case)**
2. $\lambda_{ij} = 0$ **: Indicating that Y_i will not be affected by U_j at all in open-loop mode. This loop should not be paired.**
3. $0 < \lambda_{ij} < 1$ **: Indicating that the closed-loop gain will become larger than open-loop gain when the other loops are closed. This implies that the loops are interacting and the interaction from other closed loops is smaller if RG is close to one and is larger if RG is close to zero. Avoid the pairing if $\lambda_{ij} \le 0.5$.**
4. $\lambda_{ij} > 1$ **: Indicating that the closed-loop gain will become smaller than open-loop gain when the other loops are closed. This implies that a high controller gain should be used for this pair. If some other controllers are open, this loop may become unstable. Avoid the pairing if RG is very high.**
5. $\lambda_{ij} < 0$ **: Indicating that the closed-loop gain has opposite sign of open-loop gain. This loop should not be paired.**

- **Loop pairing rule**
  - **Pair input and output variables that have positive RGA elements that are closest to 1.0.**

$$\Lambda = \begin{bmatrix} 0.8 & 0.2 \\ 0.2 & 0.8 \end{bmatrix} \quad \Lambda = \begin{bmatrix} 0.3 & 0.7 \\ 0.7 & 0.3 \end{bmatrix} \quad \Lambda = \begin{bmatrix} 1.95 & -0.65 & -0.3 \\ -0.66 & 1.88 & -0.22 \\ -0.29 & -0.23 & 1.52 \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} 2.4 & -1.4 \\ -1.4 & 2.4 \end{bmatrix} \quad \Lambda = \begin{bmatrix} -1.89 & 3.59 & -0.7 \\ -0.13 & 3.02 & -1.89 \\ 3.02 & -5.61 & 3.59 \end{bmatrix}$$

?

  - **Niederlinski Index**

$$N \equiv \frac{|\mathbf{K_p}|}{\prod_{i=1}^{n} K_{pii}}$$

For open-loop stable and 1-1/2-2/…/*n-n* configuration, the multiloop system will be unstable if the Niederlinski index is negative.

- **For 2x2 system, it is sufficient and necessary condition.**

$$N = \frac{K_{p11}K_{p22} - K_{p12}K_{p21}}{K_{p11}K_{p22}} = \frac{1}{\lambda_{11}}$$

Therefore, if the λ_{11} is negative, then 1-1/2-2 pairing is unstable.

- **For *n*x*n* systems (*n*>2), it is only sufficient condition.**

- **Example**

$$\Lambda = \begin{bmatrix} -1.89 & 3.59 & -0.7 \\ -0.13 & 3.02 & -1.89 \\ 3.02 & -5.61 & 3.59 \end{bmatrix} \quad \left(\mathbf{K} = \begin{bmatrix} 1 & 1 & -0.1 \\ 0.1 & 2 & -1 \\ -2 & -3 & 1 \end{bmatrix} \right)$$

?

– **Niederlinski index:** $N = \dfrac{|\mathbf{K}|}{K_{11}K_{22}K_{33}} = \dfrac{0.53}{1 \cdot 2 \cdot 1} > 0$

– **The 1-1/2-2/3-3 pairing may be stable, but not sure.**

– **When the first loop is open, the subsystem is unstable.**

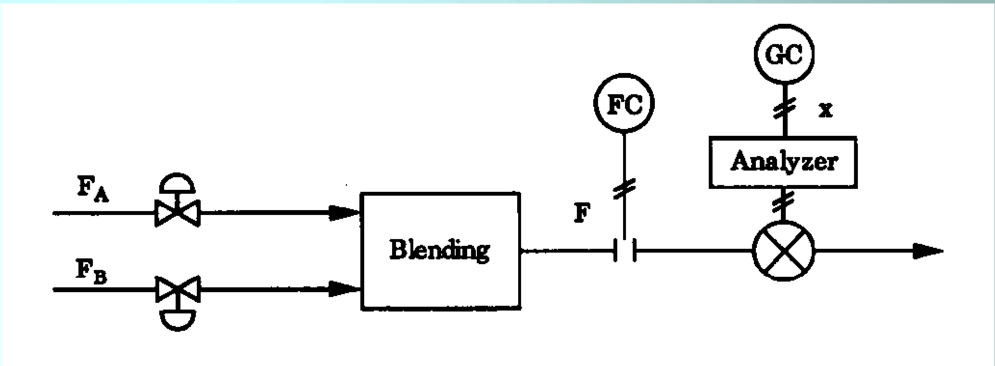
$$\mathbf{K} = \begin{bmatrix} 2 & -1 \\ -3 & 1 \end{bmatrix} \quad \Lambda = \begin{bmatrix} -2 & 3 \\ 3 & -2 \end{bmatrix} \quad N = \frac{|\mathbf{K}|}{K_{11}K_{22}} = \frac{-1}{2 \cdot 1} < 0$$

– **Such a system that is stable when all loops are closed, but that goes unstable if one of them become open is said to have a *low degree of integrity*.**

> **Always pair on positive RGA elements that are closest to 1.0 in value, and thereafter use Niederlinski's condition to check the resulting configuration for structural instability**

• Example: Blending Process

$$F = F_A + F_B \qquad x = \frac{F_A}{F_A + F_B}$$



$$\frac{\partial F}{\partial F_A} = \frac{\partial F}{\partial F_B} = 1 \qquad \frac{\partial x}{\partial F_B} = -\frac{F_A}{F^2} = -\frac{x}{F}$$

$$\frac{\partial x}{\partial F_A} = \frac{F_B}{F^2} = \frac{F - F_A}{F^2} = \frac{1-x}{F}$$

$$\begin{bmatrix} F \\ x \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ (1-x)/F & -x/F \end{bmatrix} \begin{bmatrix} F_A \\ F_B \end{bmatrix} \qquad \Lambda = \begin{bmatrix} x & 1-x \\ 1-x & x \end{bmatrix}$$

- The RGA is dependent on the product composition.
- If x is greater than 0.5, use 1-1/2-2 pairing, else choose 1-2/2-1 pairing.
- If x is close to one, F_B is very small and F_B will not affect the product flow very much, but F_B will change the composition significantly.
- This strategy implies that the larger flow of feeds is selected to control product flow and the smaller flow of feeds is selected to control composition.

- **Example: Pure integrator**

$$\mathbf{G}(s) = \begin{bmatrix} \dfrac{-1.318e^{-2.5s}}{20s+1} & \dfrac{-e^{-4s}}{3s} \\ \dfrac{0.038(182s+1)}{(27s+1)(10s+1)(6.5s+1)} & \dfrac{0.36}{s} \end{bmatrix}$$

- **Some gains become infinity.**
- **Replace $1/s$ as I and get the gains.**
- **Calculate RGA while I goes to infinity.**

$$\mathbf{K} = \begin{bmatrix} -1.318 & \dfrac{-I}{3} \\ 0.038 & 0.36I \end{bmatrix}$$

$$\lambda_{11} = \lim_{I\to\infty} \frac{1}{1+(0.038\cdot 0.333I)/(1.138\cdot 0.36I)} = 0.97 \qquad \Lambda = \begin{bmatrix} 0.97 & 0.03 \\ 0.03 & 0.97 \end{bmatrix}$$

- **1-1/2-2 pairing is recommended.**
- **If I cannot be cancelled, use other approaches suggested by McAvoy.**

- **Pairing of non-square systems**
  - **Under-defined system: more outputs than inputs**
    - **Choose same number of outputs as inputs based on the importance of the output variables.**
  - **Over-defined system: more inputs than outputs**
    - **Among possible combinations of inputs with same number of inputs as outputs, choose best subsystem based on the RGA analysis so that the subsystem has least interaction.**
- **Comments on RGA**
  - **RGA is only based on the steady-state information.**
  - **If there are some constraints on inputs, the best RGA pairing may perform poorly.**
  - **Even though the RGA analysis indicates large interaction, some processes have less interaction dynamically when the time constants are quite different.**
  - **If there are significant time delays, lags, or even inverse response, the best RGA pairing may perform poorly.**
  - **Dynamic RGA or some other modification can be used.**

Multiloop Controller Tuning

- **The multiloop controllers have some performance limitation caused by the interaction.**
  - **For highly interacting systems, the performance cannot be improved very much by controller tuning.**
- **Practical tuning method**
  - **With the other loops on manual control, tune each control loop independently for satisfaction**
  - **Then fine tune the controllers while all loops are on automatic.**
  - **Detuning method for 2x2 (McAvoy)**

$$K_{ci} = \begin{cases} \left(\lambda - \sqrt{\lambda^2 - \lambda}\right) K_{ci}^0 & \text{for } \lambda > 1.0 \\ \left|\lambda + \sqrt{\lambda^2 - \lambda}\right| K_{ci}^0 & \text{for } \lambda \le 1.0 \end{cases}$$

- **Or, use optimization method to find tuning parameters so that the performance criteria such as ITAE is minimized.**

Change of Variables

- By using transformations to create combinations of the original inputs and /or outputs of a process, it is possible to obtain an equivalent system with less interaction.

$$\mathbf{y} = \mathbf{K}\mathbf{u} \xrightarrow[\mathbf{u}=\mathbf{B}\hat{\mathbf{u}}]{\mathbf{y}=\mathbf{A}\hat{\mathbf{y}}} \hat{\mathbf{y}} = \mathbf{A}^{-1}\mathbf{K}\mathbf{B}\hat{\mathbf{u}} = \hat{\mathbf{K}}\hat{\mathbf{u}}$$

  - Find A, B, and K so that the interaction for $\hat{\mathrm{K}}$ is eliminated or at least improved .
  - Example:

$$\mathbf{K} = \begin{bmatrix} 0.878 & -0.864 \\ 1.082 & -1.096 \end{bmatrix} \quad \Lambda = \begin{bmatrix} 35.1 & -34.1 \\ -34.1 & 35.1 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 39.9 & -31.5 \\ 39.4 & -32.0 \end{bmatrix}$$

$$\Rightarrow \hat{\mathbf{K}} = \begin{bmatrix} 0.991 & -0.009 \\ -0.011 & 0.989 \end{bmatrix} \qquad \begin{pmatrix} \hat{u}_1 \\ \hat{u}_2 \end{pmatrix} = \begin{bmatrix} 0.8964u_1 - 0.8824u_2 \\ 1.1036u_1 - 1.1176u_2 \end{bmatrix}$$

- **Use of Singular Value Decomposition (SVD)**

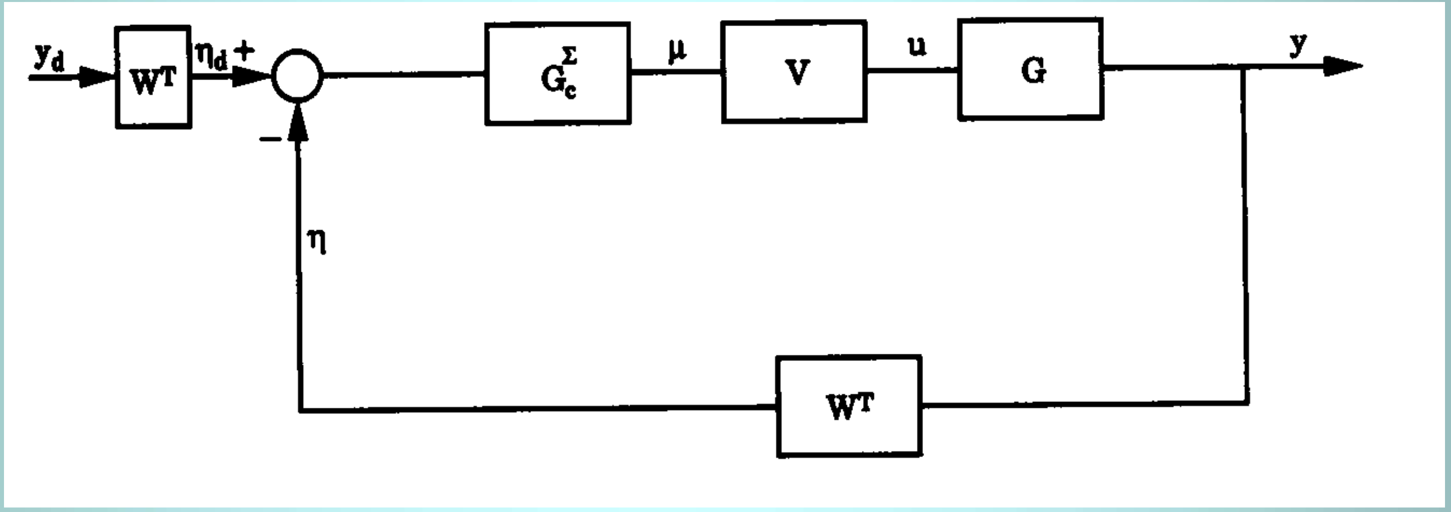
$$\mathbf{K} = \mathbf{W}\Sigma\mathbf{V}^T \quad (\Sigma\text{: diagonal matrix})$$

$$\mathbf{y} = \mathbf{W}\Sigma\mathbf{V}^T\mathbf{u}$$

Let $\eta = \mathbf{W}^T\mathbf{y}$ and $\mu = \mathbf{V}^T\mathbf{u}$.

Then $\eta = \Sigma\ \mu$ (completely decoupled at steady state)

  - **The controller should be designed based on new transformed inputs and outputs.**



Decoupling Control

- **The design objective**
  - **The reduction of control loop interactions by adding additional controllers called *decouplers* to a conventional multiloop control configuration.**
- **Theoretical benefits**
  - **Control loop interaction are eliminated and the stability of the closed-loop system is determined by the stability characteristics of the individual feedback control loops**
  - **A set-point change for one CV has no effect on the other CV's**
- **In practice**
  - **Reduction of control loop interactions (not a perfect elimination of interactions due to the imperfect process model and the physical realizability of the decouplers)**

- **Decoupler Design**
  - **The effect of u_1 on y_2 through G_{p21} can be cancelled by using a decoupler D_{21} going through G_{p22}.**

$$G_{p21}U_1 = -D_{21}G_{p22}U_1 \Rightarrow D_{21} = -G_{p21} / G_{p22}$$

  - **In the same manner,**

$$G_{p12}U_2 = -D_{12}G_{p11}U_2 \Rightarrow D_{12} = -G_{p12} / G_{p11}$$

  - **Ideal decoupler**
    - **Similar to a FF controller**
    - **May be *unstable* or *physically unrealizable***
    - **Often implemented as a lead-lag module or a static decoupler**

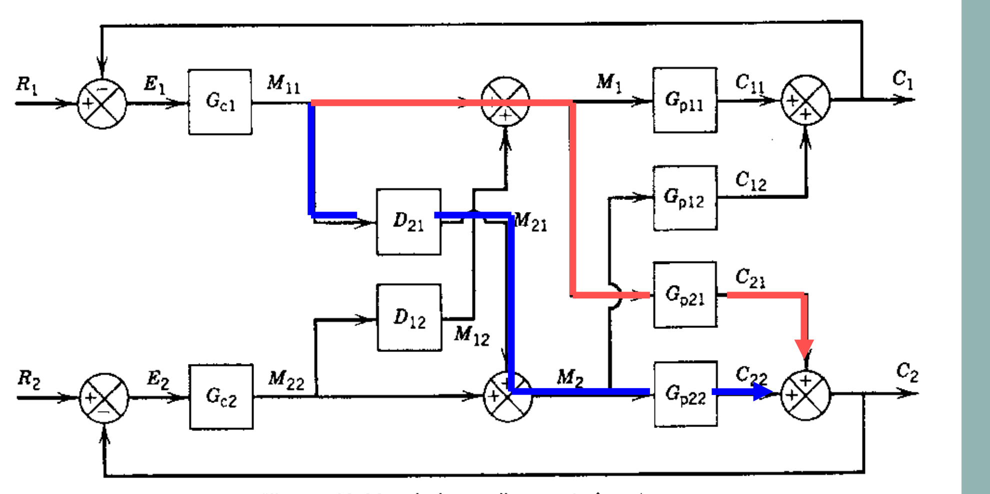


Figure 19.11. A decoupling control system.

- **General Case Design**

$\mathbf{y} = \mathbf{G}\mathbf{u}$ ($\mathbf{G}$ is a full matrix) $\Rightarrow \mathbf{y} = \mathbf{G}_D\mathbf{v}$ ($\mathbf{G}_D$ is diagonal)

$\mathbf{u} = \mathbf{D}\mathbf{v}$ ($\mathbf{D}$ is the decoupler)

$\therefore \mathbf{G}_D = \mathbf{G}\mathbf{D} \Rightarrow \mathbf{D} = \mathbf{G}^{-1}\mathbf{G}_D$

A common choice for $\mathbf{G}_D = diag(\mathbf{G})$

- **2x2 case**

$$\text{Let } \mathbf{D} = \begin{bmatrix} 1 & D_{12} \\ D_{21} & 1 \end{bmatrix}, \quad \mathbf{G}_D = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}\begin{bmatrix} 1 & D_{12} \\ D_{21} & 1 \end{bmatrix} = \begin{bmatrix} G_{11} + D_{21}G_{12} & G_{12} + D_{12}G_{11} \\ G_{21} + D_{21}G_{22} & G_{22} + D_{12}G_{21} \end{bmatrix}$$

$$\therefore D_{12} = -G_{12}/G_{11} \text{ and } D_{21} = -G_{21}/G_{22}$$

- **3x3 cases**

$$\text{Let } \mathbf{D} = \begin{bmatrix} 1 & D_{12} & D_{13} \\ D_{21} & 1 & D_{23} \\ D_{31} & D_{32} & 1 \end{bmatrix}, \text{ and solve for } D_{ij}.$$

- **Example**

$$\mathbf{G_p} = \begin{bmatrix} \dfrac{5e^{-5s}}{4s+1} & \dfrac{2e^{-4s}}{8s+1} \\ \dfrac{3e^{-3s}}{12s+1} & \dfrac{6e^{-3s}}{10s+1} \end{bmatrix}$$

$$D_{21} = -\frac{G_{p21}}{G_{p22}} = -\frac{0.5(10s+1)}{12s+1}$$

$$D_{12} = -\frac{G_{p12}}{G_{p11}} = -\frac{0.25(4s+1)e^{s}}{8s+1}$$

- **$D_{21}(s)$ has a pole and zero which are very close each other. Thus, $D_{21}(s)$ =-0.5 is a quite reasonable approximation.**
- **$D_{12}(s)$ has time lead instead of time delay, which is physically unrealizable.**
    - **Time lead is 1 which is relatively small compared to time constants. Thus, neglect time lead and use a lead-lag type.**
- **Due to the modeling error of the process, the perfect decoupling would not be possible anyway.**
- **In many cases, the steady-state decouplers will be beneficial to reduce the control loop interactions.**

Alternative Decoupling Control System

- **The original configurations**
  - The decoupler uses the controller output signal which may be different from actual input to process due to saturation.
  - It may cause the wind up.
- **New approach**
  - Use same input to process
  - It is more sensitive to modeling error

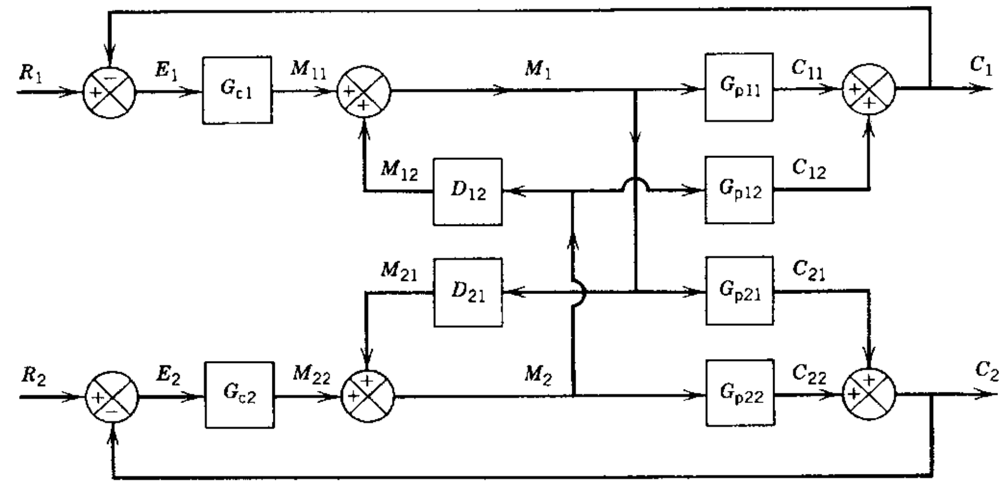


Figure 19.12. An alternative decoupling control scheme.

- **Experimental application to distillation column**
  - **Outperforms the conventional multiloop PI control**

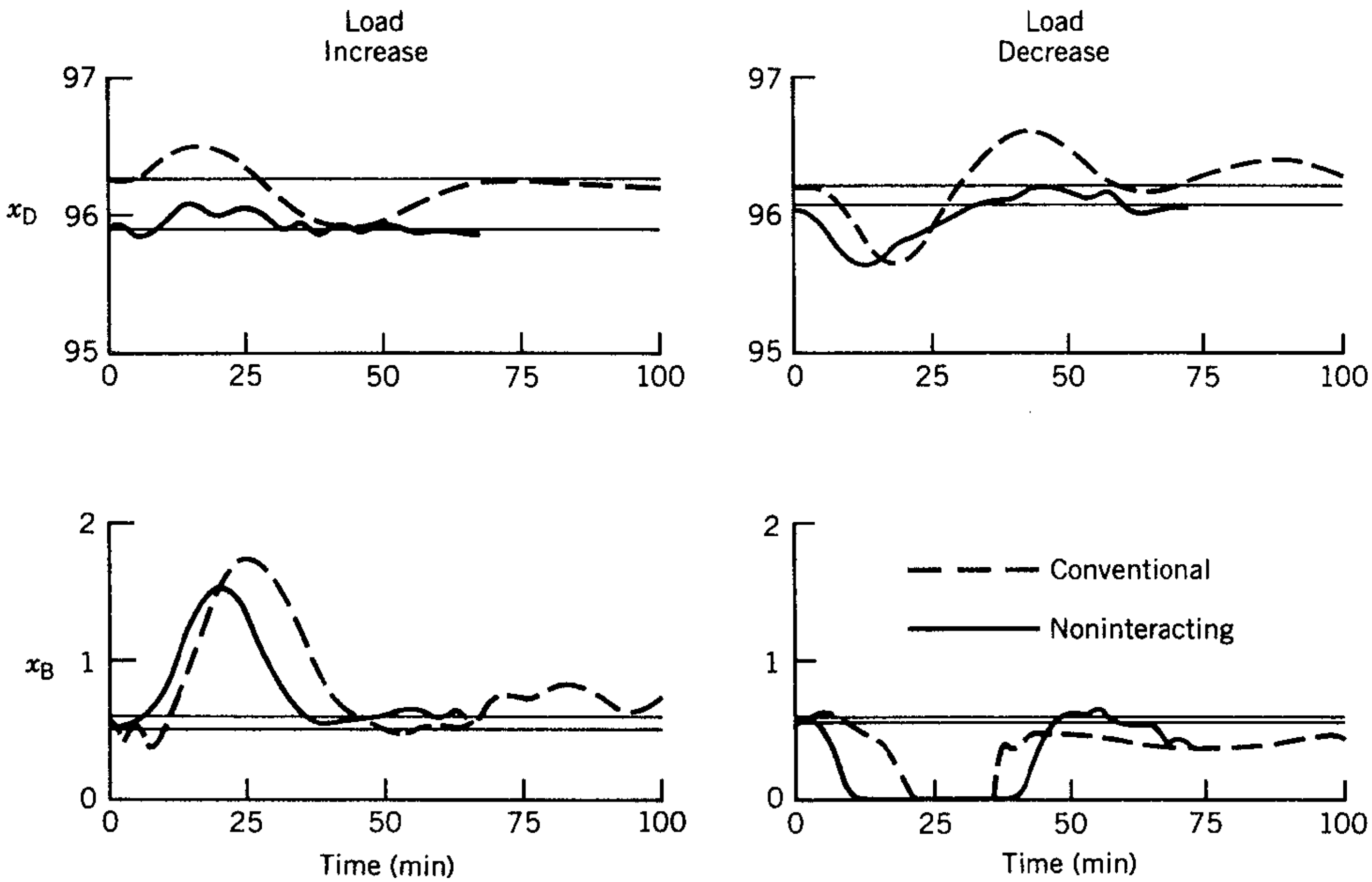


Figure 19.13. An experimental application of decoupling (noninteracting) control to a distillation column [3].

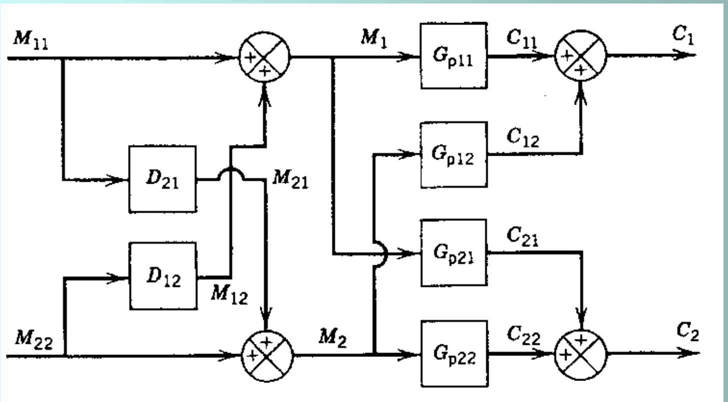
Other types of decoupling

- **Partial (one-way) Decoupling**
  - **Set some of decouplers zero.**
  - **This is very attractive if one CV is more important than the others, or one interaction is much weaker than the others or absent.**
  - ***Less sensitive*** **to modeling errors**
  - **The partial decoupling can provide better control than the complete decoupling in some situations.**

- **Nonlinear decouplers**
  - **If the process is nonlinear or time-varying, the linear decoupler would be worse than conventional multiloop PID schemes.**
  - **Then, the nonlinear decoupler can be considered.**

Sensitivity of the decouplers

- **For the imperfect model (static case)**



$$\frac{y_1}{u_{11}} = D_{21}K_{p12} + K_{p11} = K_{p11} - K_{p12}\frac{\tilde{K}_{p21}}{\tilde{K}_{p22}}$$

$$\frac{y_2}{u_{11}} = D_{21}K_{p22} + K_{p21} = K_{p21}\left(1 - \frac{\tilde{K}_{p21}}{K_{p21}}\frac{K_{p22}}{\tilde{K}_{p22}}\right)$$

$$\frac{y_1}{u_{22}} = K_{p12}\left(1 - \frac{\tilde{K}_{p12}}{K_{p12}}\frac{K_{p11}}{\tilde{K}_{p11}}\right) \qquad \frac{y_2}{u_{22}} = K_{p22} - K_{p21}\frac{\tilde{K}_{p12}}{\tilde{K}_{p11}} = K_{p22}/\lambda_{11} \text{ (if no error)}$$

Let $\kappa = K_{p12}K_{p21}/K_{p11}K_{p22}$ and $e_{ij} = \tilde{K}_{pij}/K_{pij}$

$$\tilde{\lambda}_{11} = \frac{1}{1 - \kappa\dfrac{(e_{11}-e_{12})(e_{22}-e_{21})}{(e_{11}-\kappa e_{12})(e_{22}-\kappa e_{21})}}$$

If the RG is large, the decoupled process gain becomes very small and large controller gain should be used. (It may cause trouble if there is model error.)

- **If there is no modeling error ($e_{ij}=1$)**
  - **Regardless of κ, $\lambda = 1$ (no interaction)**
- **If there is no interaction, ($\kappa = 0$)**
  - **No effect by modeling error ($\lambda = 1$)**
- **If there are large interaction ($\kappa \to 1$)**
  - **Still large interaction even with decoupler**

$$\lambda_{11} = \frac{1}{1-\kappa\dfrac{(e_{11}-e_{12})(e_{22}-e_{21})}{(e_{11}-\kappa e_{12})(e_{22}-\kappa e_{21})}} = \frac{1}{1-\kappa} \to \infty$$

  - **The RG becomes unity only when e_{ij} are ones.**

- **Thus, the high RGA processes may have strong sensitivity to modeling errors.**

- **Analysis in vector-matrix form (steady state)**

$$\mathbf{y} = \mathbf{G}\mathbf{u},\ \ \mathbf{u} = \mathbf{D}\mathbf{v},\ \text{and}\ \ \mathbf{D} = \tilde{\mathbf{G}}^{-1}\mathbf{G}_D$$

$$\mathbf{y} = \mathbf{G}\tilde{\mathbf{G}}^{-1}\mathbf{G}_D\mathbf{v} \Rightarrow \mathbf{y} = \mathbf{K}\tilde{\mathbf{K}}^{-1}\mathbf{K}_D\mathbf{v}$$

$$\text{Let } \mathbf{K} = \tilde{\mathbf{K}} + \Delta\tilde{\mathbf{K}}.$$

$$\mathbf{y} = (\tilde{\mathbf{K}} + \Delta\tilde{\mathbf{K}})\tilde{\mathbf{K}}^{-1}\mathbf{K}_D\mathbf{v} \Rightarrow \Delta\mathbf{y} = \Delta\tilde{\mathbf{K}}\tilde{\mathbf{K}}^{-1}\mathbf{K}_D\mathbf{v}$$

$$\therefore \Delta\mathbf{y} = \frac{\Delta\tilde{\mathbf{K}}Adj(\mathbf{K})\mathbf{K}_D\mathbf{v}}{|\mathbf{K}|}$$

- **If the determinant of K is small**
  - **Small modeling errors will be magnified into very large error in y.**
  - **Small change in controller output v will also result in large error in y.**
  - **If the *determinant is zero*, then some outputs are dependent each other and *independent control is impossible*. (degeneracy)**

Less obvious ill-conditioned case

- **Example 1**

$$K = \begin{bmatrix} -60 & 0.05 \\ -40 & -0.05 \end{bmatrix} \quad \det(\mathbf{K}) = 0.05(60+40) = 5 \quad \lambda_{11} = 1/(1+40/60) = 0.6$$

$$\left|\frac{\lambda_{\max}}{\lambda_{\min}}\right| = \frac{59.965}{0.0835} \approx 720 \text{ (condition number)}$$

  - **No unusual indicator**
  - **But effect of u_1 is much greater than u_2.**

- **Example 2**

$$K = \begin{bmatrix} -3 & 1 \\ -2 & -1 \end{bmatrix} \quad \det(\mathbf{K}) = 1(3+2) = 5 \quad \lambda_{11} = 1/(1+2/3) = 0.6$$

$$\left|\frac{\lambda_{\max}}{\lambda_{\min}}\right| = \left|\frac{-2+j}{-2-j}\right| = 1 \text{ (condition number)}$$

  - **Not ill-conditioned from eigenvalues**
  - **Easy to decouple**

- **Example 3**

$$K = \begin{bmatrix} 1 & 0.001 \\ 100 & 1 \end{bmatrix} \quad \det(\mathbf{K}) = 1 - 0.1 = 0.9 \quad \lambda_{11} = 1/(1-0.1/1) = 1.11$$

$$\left|\frac{\lambda_{\max}}{\lambda_{\min}}\right| = \left|\frac{1.316}{0.684}\right| = 1.924 \text{ (condition number)}$$

  - **Not ill-conditioned from eigenvalues**
  - **But effect of u_1 is much greater than u_2.**

- **Most reliable indicator of the interaction**
  - **Determinant of K, RGA, or condition number cannot be an reliable indicator of the ill conditioning in a matrix.**
  - ***Singular value*: eigenvalue of the matrix K^TK**
  - **The *condition number based on singular values* is the most reliable indicator of the matrix condition.**

$$K = \begin{bmatrix} 1 & 0.001 \\ 100 & 1 \end{bmatrix} \qquad \left|\frac{\sigma_{\max}}{\sigma_{\min}}\right| = \left|\frac{100.01}{0.009}\right| = 1.113 \times 10^4 \text{ (condition number)}$$

- **Conclusion**

  Feasibility of decoupling is directly related to the conditioning of the process gain matrix. Decoupling is only feasible to the degree that the process is well conditioned; it is virtually impossible to achieve decoupling in a poorly conditioned process.