Part I

Linear Algebra

Chapter 1

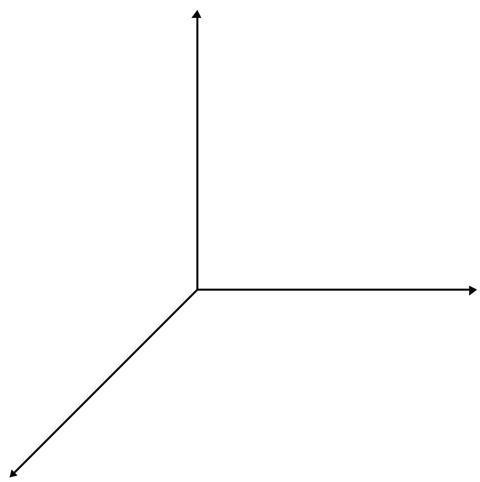
Vectors

1.1 Scalars and Vectors

Scalar: a quantity described by a single number that has magnitude only.

Vector: a quantity described by a directed line segment that has magnitude and direction.

Cartesian (rectangular) coordinate system: three mutually perpendicular straight lines:



On each line, consider the unit point whose distance from the origin is 1. The vectors defined by the directed line from the origin to the unit point are denoted by

$$i = e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad j = e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad k = e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Representation of Vector in a Coordinate System:

Given a coordinate system, a vector a can be represented by triples of real numbers:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 e_1 + x_2 e_2 + x_3 e_3$$

Given a coordinate system, the vector and its representation are identified.

Warning: a vector has different representation in different coordinate systems.

Zero (Null) Vector: $0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Magnitudes (Norm) of Vectors:

p norms:

$$||x||_p = (|x_1|^p + |x_2|^2 + |x_3|^p)^{\frac{1}{p}} \quad 1 \leq p < \infty$$

$$||x||_\infty = \max\{|x_1|, |x_2|, |x_3|\}$$

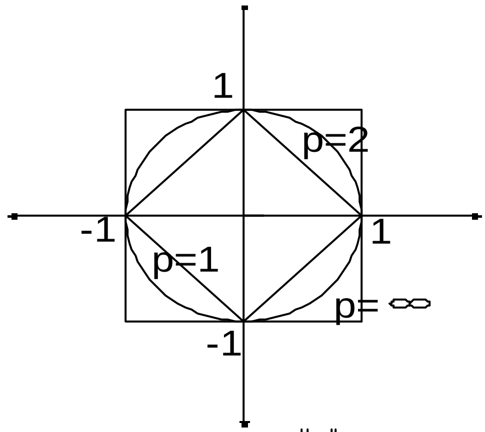
Example:

$$||x||_1 = |x_1| + |x_2| + |x_3|$$

$$||x||_2 = \sqrt{|x_1|^2 + |x_2|^2 + |x_3|^2}$$

$$||x||_\infty = \max\{|x_1|, |x_2|, |x_3|\}$$

$||x||_2$ coincides with the length in Euclidean sense and, thus, is called Eclidean norm. Throughout the lecture, we use the Euclidean norm unless stated otherwise and $||\cdot||$ denotes $||\cdot||_2$.



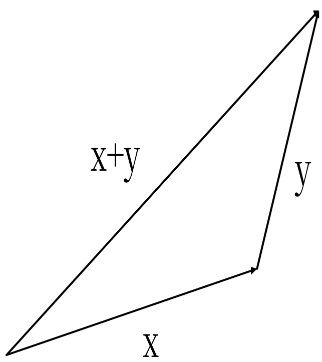
Unit Ball $\{x : \|x\|_p \le 1\}$

Basic Algebraic Operations of Vectors

α: a scalar, x, y: vectors

Addition:

$$x + y = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{bmatrix}$$



Scalar Multiplication:

$$ax = a \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} ax_1 \\ ax_2 \\ ax_3 \end{bmatrix}$$

Properties of Addition:

- Commutativity: $x + y = y + x$
- Associativity: $(x + y) + z = x + (y + z)$
- $x + 0 = 0 + x = x$
- $x + (-x) = 0$

Properties of Scalar Multiplication:

- $||ax|| = |a|||x||$
- Distributivity: $a(x + y) = ax + ay$
- Distributivity: $(a + b)x = ax + bx$
- Associativity: $a(bx) = (ab)x$
- $1x = x$
- $0x = 0$

1.2 Linear Independence

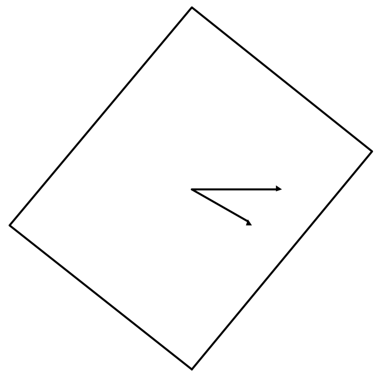
Linear (or Vector) Space $\mathbf{R}^n$: set of vectors

Linear Combination: given $\{x_1, \cdots, x_m\}$,

$$a_1x_1 + a_2x_2 + \cdots + a_mx_m, \quad a_i \in \mathbf{R}.$$

Span: Span of $x_1, \cdots, x_m$ is the set of all linear combination of them, which is a plane in $\mathbf{R}^n$.

$$span\{x_1, x_2, \cdots, x_m\} = \{x = a_1x_1 + a_2x_2 + \cdots + a_mx_m, \ a_i \in \mathbf{R}\}$$



Linear Independence: $\{x_1, \cdots, x_m\}$ is called linearly independent if no one of them is in the span of others:

$$x_i \notin span\{x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_m\}, \quad \forall i.$$

Theorem: The family of vectors $\{v_i\}_{i=1}^m \subset \mathbf{R}^n$ are linear dependent iff $\exists$ $\{a_i\}_{i=1}^m$, not all zero such that

$$a_1 v_1 + \cdots + a_m v_m = 0.$$

Proof: $(\Leftarrow)$ Linear dependence implies there exists i such that

$$x_i \in span\{x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_m\}.$$

$$\Downarrow$$

There exists a_i's such that

$$x_i = a_1 x_1 + \cdots + a_{i-1} x_{i-1} + a_{i+1} x_{i+1} + \cdots + a_m x_m.$$

$$\Downarrow$$

$$a_1 x_1 + \cdots + a_{i-1} x_{i-1} - x_i + a_{i+1} x_{i+1} + \cdots + a_m x_m = 0.$$

($\Rightarrow$) Let $a_i \neq 0$. Then

$$x_i = -\frac{a_1}{a_i}x_1 - \cdots - \frac{a_{i-1}}{a_i}x_{i-1} - \frac{a_{i+1}}{a_i}x_{i+1} - \cdots - \frac{a_m}{a_i}x_m.$$

Hence $\{v_i\}_{i=1}^m$ is linearly dependent.

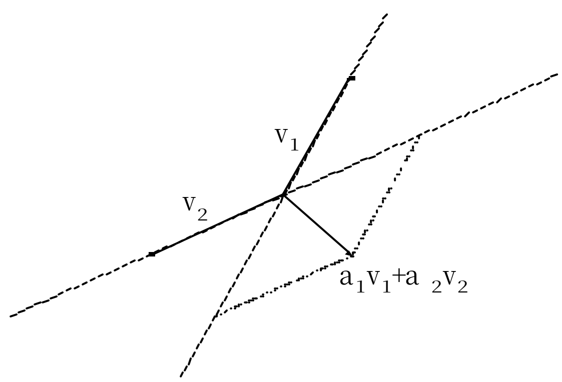
By contraposition, we get the following corollary.

Corollary: The family of vectors $\{v_i\}_{i=1}^m \subset \mathbf{R}^n$ are linear independent iff

$$a_1 v_1 + \cdots + a_m v_m = 0 \quad \Rightarrow \quad a_i = 0 \ \forall i.$$

Def.: The maximal number of linearly independent vectors in a linear space is called the dimension of the linear space.

Ex: $dim(\mathbf{R}^n) = n$



Definition: the family of vectors $\{b_i\}_{i=1}^n \subset \mathbf{R}^n$ is said to be a basis if elements of the family are linearly independent each other and

$$V = Span(\{b_i\}_{i=1}^n).$$

The elements of the family are called basis vectors of $\mathbf{R}^n$.

Note: If $x \in \mathbf{R}^n$, $\exists$ unique $\{\xi_i\}_{i=1}^n$ such that

$$x = \sum_{i=1}^{n} \xi_i b_i$$

$\xi = \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix}$ is called the component vector (or representation) of x w.r.t. the basis $\{b_i\}_{i=1}^n$.

Theorem: In a n-dimensional vector space, any set of n linearly independent vectors qualifies as a basis.

Proof: Let $\{u_i\}_{i=1}^n$ be a set of n linearly independent vectors. Then $\{x\} \cup \{u_i\}_{i=1}^n$ is linearly dependent. Hence

$$a_0 x + a_1 u_1 + \cdots + a_n u_n = 0$$

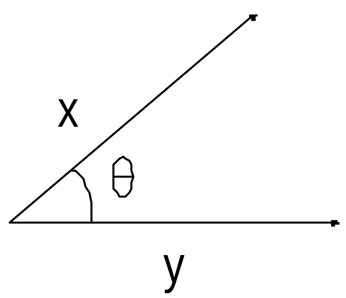
where not all a_i's are zero. This implies $a_0 \neq 0$ since $\{u_i\}_{i=1}^n$ is linearly dependent otherwise. Then

$$x = b_1 u_1 + \cdots + b_n u_n$$

where $b_i = -\frac{a_i}{a_0}$.

1.3 Inner Product

Inner (Dot or Scalar) Product:



$$x \cdot y = ||x|| ||y|| \cos\theta$$

$$\Downarrow$$

$$x \cdot y \begin{cases} > 0 & \text{if } \theta \text{ is acute} \\ = 0 & \text{if } \theta \text{ is right} \\ < 0 & \text{if } \theta \text{ is obtuse} \end{cases}$$

Hence, two vectors x, y are orthogonal if $x \cdot y = 0$.
Properties of Inner Product:

- $x \cdot x = ||x||^2 \Rightarrow ||x|| = \sqrt{x \cdot x}$.
- $\cos(\theta) = \frac{x \cdot y}{||x|| ||y||} = \frac{x \cdot y}{\sqrt{x \cdot x}\sqrt{y \cdot y}}$
- Linearity: $(a_1 x + a_2 y) \cdot z = a_1 x \cdot z + a_2 y \cdot z$
- Symmetry: $x \cdot y = y \cdot x$
- Positive Definiteness: $x \cdot x > 0$ for all $x \neq 0$.
- Schwarz inequality: $||x \cdot y|| \leq ||x|| ||y||$

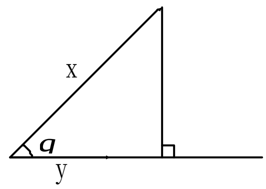
Important results that can be proven using inner product:

- Triangular inequality: $||x + y|| \leq ||x|| + ||y||$
- Parallelogram equality: $||x + y||^2 + ||x - y||^2 = 2(||x||^2 + ||y||^2)$

Clearly $e_i \cdot e_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$. Therefore

$$x \cdot y = x_1 y_1 + x_2 y_2 + x_3 y_3 =: x^T y.$$

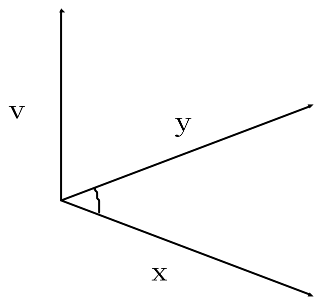
Projection (component) of x in the direction of y: $p = ||x|| \cos\theta = \frac{x \cdot y}{||y||}$.



Notice that $|p|$ is the length of orthogonal projection of x on a straight line in the direction y.

1.4 Outer Product

Outer (cross or vector) Product: $v = x \times y$ is a vector whose magnitude is the area of the parallelogram formed by x and y and whose direction is the direction to which as right-handed screw advances as the screw is turned from x to y.



Notice that

$$||v|| = ||x|| ||y|| \sin\theta.$$

Properties of cross product:

- $$i \times i = j \times j = k \times k = 0$$
  $$i \times j = k, \quad j \times k = i, \quad k \times i = j$$
- Anticommutativity: $y \times x = -x \times y$
- Not associative: $x \times (y \times z) \neq (x \times y) \times z$
  Ex: $i \times (i \times j) = i \times k = -j \neq 0 = 0 \times j = (i \times i) \times j$.
- $(ax) \times y = a(x \times y) = x \times (ay)$
- $x \times (y + z) = (x \times y) + (x \times z)$ and $(x + y) \times z = (x \times z) + (y \times z)$

$$x \times y = (x_1 i + x_2 j + x_3 k) \times (y_1 i + y_2 j + y_3 k)$$
$$= (x_2y_3 - x_3y_2)i + (x_3y_1 - x_1y_3)j + (x_1y_2 - x_2y_1)k = det \begin{bmatrix} i & j & k \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}$$

Fact: Two vectors are linearly dependent if their cross product is zero vector.

1.5 Triple Products

Scalar Triple Product: $x \cdot (y \times z)$

$$x \cdot (y \times z) = x_1(y_2z_3 - y_3z_2) + x_2(y_3z_1 - y_1z_3) + x_3(y_1z_2 - y_2z_1) = det \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$$

Since interchanging of two rows reverses the sign of the determinant,

$$x \cdot (y \times z) = y \cdot (z \times x) = z \cdot (x \times y) = -x \cdot (z \times y) = -z \cdot (y \times x) = -y \cdot (x \times z)$$

Hence,
$$x \cdot (y \times z) = (x \times y) \cdot z$$

Moreover, it is easy to see that

$$(ax) \cdot (y \times z) = a[x \cdot (y \times z)]$$

Fact: Three vectors form a linearly dependent set iff there scalar triple product is zero.

Chapter 2

Matrices

2.1 Linear Operators

A: an operator (or transformation or mapping) from $\mathbf{R}^n$ to $\mathbf{R}^m$

$$\underbrace{y}_{\mathbf{R}^m} = A \underbrace{x}_{\mathbf{R}^n}.$$

Def.: A is linear if

$$A(a_1u_1 + a_2u_2) = a_1Au_1 + a_2Au_2, \quad u_1, u_2 \in \mathbf{R}^n, \quad a_1, a_2 \in \mathbf{R}.$$

Null Space (Kernel):

$$\mathcal{N}(A) = \{u \in \mathbf{R}^n : Au = 0\}$$

Range Space (Image):

$$\mathcal{R}(A) = \{v \in \mathbf{R}^m : v = Au, u \in \mathbf{R}^n\} = A\mathbf{R}^n$$

Fact: $0 \in \mathcal{N}(A)$.
Proof: $A0 = A(0 \cdot u) = 0Au = 0$.
Fact: $\mathcal{N}(A)$ and $\mathcal{R}(A)$ are linear subspaces.
Proof: Let $x_1, x_2 \in \mathcal{N}(A)$. Then $Ax_1 = Ax_2 = 0$. By linearity,

$$A(ax_1 + bx_2) = aAx_1 + bAx_2 = 0$$

$\Rightarrow$ $ax_1 + bx_2 \in \mathcal{N}(A)$.

Let $y_1, y_2 \in \mathcal{R}(A)$. Then $\exists x_1, x_2 \in \mathbf{R}^n$ such that $y_1 = Ax_1$ and $y_2 = Ax_2$. Then $ax_1 + bx_2 \in \mathbf{R}^n$. By linearity,

$$A(ax_1 + bx_2) = aAx_1 + bAx_2 = ay_1 + by_2 \in \mathcal{R}(A)$$

Theorem: A is injective (one-to-one) iff $\mathcal{N}(A) = \{0\}$.
Proof: ($\Rightarrow$) Obvious
($\Leftarrow$) Suppose the contrary. Then $\exists\ x \neq y$ such that $Ay = Ax \Rightarrow A(y - x) = 0 \Rightarrow y = x$ (contradiction).
Facts:

1. If $\{Au_i\}$ is a linearly independent family, then so is $\{u_i\}$.

2. The converse of the above holds if A is injective.

Proof: 1)

$$a_1u_1 + \cdots + a_nu_n = 0$$

$$\Downarrow$$

$$a_1Au_1 + \cdots + a_nAu_n = A(a_1u_1 + \cdots + a_nu_n) = 0$$

$$\Downarrow$$

$$a_1 = \cdots = a_n = 0$$

2) ($\Leftarrow$)

$$0 = a_1Au_1 + \cdots + a_nAu_n = A(a_1u_1 + \cdots + a_nu_n)$$

By one-to-one assumption,

$$a_1u_1 + \cdots + a_nu_n = 0$$

$$\Downarrow$$

$$a_1 = \cdots = a_n = 0$$

Theorem: Suppose A be a linear operator from $\mathbf{R}^n$ to $\mathbf{R}^n$. Then TFAE

1. A is injective ($\mathcal{N}(A) = \{0\}$)

2. A is surjective (onto) ($\mathcal{R}(A) = V$)

3. A is bijective (injective+surjective)

4. A^{-1} exists

Proof: ($1 \Rightarrow 2$) From the previous Fact 2), $\{u_i\}$ is a basis implies $\{Au_i\}$ is so.

$$\mathbf{R}^n = \{a_1 Au_i + \cdots + a_n Au_n : a_i \in \mathbf{R}\} = \{A\underbrace{(a_1 u_i + \cdots + a_n u_n)}_{\in \mathbf{R}^n} : a_i \in \mathbf{R}\}$$

$$= \{Au : u \in \mathbf{R}^n\} = \mathcal{R}(A).$$

($1 \Leftarrow 2$) Let $\{v_i\}$ be a basis of $\mathbf{R}^n$.

A surjective $\Rightarrow \exists\ u_i$ such that $v_i = Au_i \Rightarrow \{u_i\}$ is a basis of $\mathbf{R}^n$.

Consider $u \neq 0$. Then $u = a_1 u_1 + \cdots + a_n u_n \neq 0$ where a_i's are not all zero. Hence, $Au = A(a_1 u_1 + \cdots + a_n u_n) = a_1 \underbrace{Au_1}_{v_1} + \cdots + a_n \underbrace{Au_n}_{v_n} \neq 0$ and, thus, $\mathcal{N}(A) = \{0\}$.

($2 \Leftrightarrow 3$) Obvious from the equivalence of 1) and 2).

($3 \Rightarrow 4$) inverse mapping is well defined.

($3 \Leftarrow 4$) Notice that, if A^{-1} exists, $Ax_1 = Ax_2$ implies $x_1 = x_2$. Hence, injective.

2.2 Matrices

An $n \times m$ matrix is a rectangular array of numbers:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$$

Transpose of a Matrix A:

$$A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}$$

Conjugate Transpose of a Matrix $\boldsymbol{A}$:

$$\boldsymbol{A}^* = \begin{bmatrix} \bar{a}_{11} & \bar{a}_{21} & \cdots & \bar{a}_{m1} \\ \bar{a}_{12} & \bar{a}_{22} & \cdots & \bar{a}_{m2} \\ \vdots & \vdots & \vdots & \vdots \\ \bar{a}_{1n} & \bar{a}_{2n} & \cdots & \bar{a}_{mn} \end{bmatrix}$$

Notice that $\boldsymbol{A}^T = \boldsymbol{A}^*$ for real matrices.

Basic Algebraic Operation of Matrices

α: a scalar, $\boldsymbol{A}, \boldsymbol{B}$: matrices

Addition:

$$\boldsymbol{A}+\boldsymbol{B} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots & a_{mn}+b_{mn} \end{bmatrix}$$

Scalar Multiplication:

$$\alpha\boldsymbol{A} = \alpha \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} \alpha a_{11} & \alpha a_{12} & \cdots & \alpha a_{1n} \\ \alpha a_{21} & \alpha a_{22} & \cdots & \alpha a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \alpha a_{m1} & \alpha a_{m2} & \cdots & \alpha a_{mn} \end{bmatrix}$$

Matrix Multiplication:

$$\boldsymbol{AB} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1l} \\ b_{21} & b_{22} & \cdots & b_{2l} \\ \vdots & \vdots & \vdots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nl} \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{i=1}^{n} a_{1i}b_{i1} & \sum_{i=1}^{n} a_{1i}b_{i2} & \cdots & \sum_{i=1}^{n} a_{1i}b_{il} \\ \sum_{i=1}^{n} a_{2i}b_{i1} & \sum_{i=1}^{n} a_{2i}b_{i2} & \cdots & \sum_{i=1}^{n} a_{2i}b_{il} \\ \vdots & \vdots & \vdots & \vdots \\ \sum_{i=1}^{n} a_{mi}b_{i1} & \sum_{i=1}^{n} a_{mi}b_{i2} & \cdots & \sum_{i=1}^{n} a_{mi}b_{il} \end{bmatrix}$$

2.3 Matrix Representation of Linear Operators

Let $\{u_j\}_{j=1}^{n}$ be the basis for $\mathbf{R}^n$. Then

$$x = \sum_{j=1}^{n} \xi_j u_j$$

By linearity of A,

$$Ax = A\sum_{j=1}^{n} \xi_j u_j = \sum_{j=1}^{n} \xi_j A u_j$$

Let $\{v_i\}_{i=1}^{m}$ be the basis for $\mathbf{R}^m$. Then

$$Au_j = \sum_{i=1}^{m} a_{ij} v_i$$

$$\Downarrow$$

$$\sum_{i=1}^{m} \eta_i v_i = y = Ax = \sum_{j=1}^{n} \xi_j A u_j = \sum_{j=1}^{n} \xi_j \sum_{i=1}^{m} a_{ij} v_i = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} \xi_j \right) v_i$$

By uniqueness of representation,

$$\eta = A\xi$$

where

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

Theorem: Let $\{u_j\}_{j=1}^{n}$ and $\{v_i\}_{i=1}^{m}$ be the bases for $\mathbf{R}^n$ and $\mathbf{R}^m$, respectively. Then, w.r.t. these bases, A is represented by the $m \times n$ matrix.

2.4 Rank and Nullity

Fact: Let $A = [c_1 \ \cdots \ c_n]$ where c_i is the ith column of A. Then $\mathcal{R}(A) = span(\{c_j\})$.

Ex: Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$. Then

$$\mathcal{R}(A) = \left\{ y = Ax = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\}$$
$$= Span \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\}.$$

Definition: The rank (nullity) of the $m \times n$ matrix $A = dim \mathcal{R}(A)$ $(dim \mathcal{N}(A))$.

Fact: $rank(A) + nullity(A) = n = dim \mathcal{D}(A)$

Proof: Let $\{u_i\}_{i=1}^k$ be the basis of $\mathcal{N}(A)$. Complete that basis such that $\{u_i\}_{i=1}^n$ is the basis of $\mathbf{R}^n$. Then $x = \sum_{i=1}^n \xi_i u_i$ and

$$Ax = A\left(\sum_{i=1}^n \xi_i u_i\right) = \sum_{i=1}^k \xi_i Au_i + \sum_{i=k+1}^n \xi_i Au_i$$

$Au_i = 0$, $i = 1, \cdots, k$ because $u_i \in \mathcal{N}(A)$.

$\Rightarrow \{Au_i\}_{i=k+1}^n$ spans $\mathcal{R}(A)$.

Claim: $\{Au_i\}_{i=k+1}^n$ is a linearly independent family. Assume the contrary. Then $\exists\ a_{k+1}, \cdots, a_n$ (not all zero) such that

$$0 = \sum_{i=k+1}^n a_i Au_i = A\left(\sum_{i=k+1}^n a_i u_i\right).$$

$\Rightarrow \sum_{i=k+1}^n a_i u_i \in \mathcal{N}(A) \Rightarrow$ contradiction and the claim follows.

$\{Au_i\}_{i=k+1}^n$ is a basis for $\mathcal{R}(A)$ with $dim \mathcal{R}(A) = n - k$.

One may conjecture that $\mathcal{N}(A)$ and $\mathcal{R}(A)$ are disjoint. But this is not the case.

Ex: $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, then

$$\mathcal{N}(A) = \left\{ 0 = Ax = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ 0 \end{bmatrix} \right\} = Span \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$$

$$\mathcal{R}(A) = \left\{ y = Ax = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ 0 \end{bmatrix} \right\} = Span \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}.$$

Theorem: A and A^T have the same rank.

Proof: Let $c_1, \cdots, c_n$ be the columns of A. Suppose $b_1, \cdots, b_r$ be the basis vectors of $\mathcal{R}(A)$ where $r = rank(A)$. Then

$$c_1 = d_{11}b_1 + \cdots + d_{1k}b_k$$

$$c_2 = d_{21}b_1 + \cdots + d_{2k}b_k$$

$$\vdots$$

$$c_n = d_{n1}b_1 + \cdots + d_{nk}b_k.$$

This implies

$$\begin{bmatrix} a_{j1} \\ a_{j2} \\ \vdots \\ a_{jn} \end{bmatrix} = \begin{bmatrix} d_{11}b_{1j} + \cdots + d_{1k}b_{kj} \\ d_{21}b_{1j} + \cdots + d_{2k}b_{kj} \\ \vdots \\ d_{n1}b_{1j} + \cdots + d_{nk}b_{kj} \end{bmatrix} = b_{1j} \begin{bmatrix} d_{11} \\ d_{21} \\ \vdots \\ d_{n1} \end{bmatrix} + \cdots + b_{kj} \begin{bmatrix} d_{1k} \\ d_{2k} \\ \vdots \\ d_{nk} \end{bmatrix}$$

Hence, the columns of A^T are in the span of the k vectors on RHS and thus $rank(A^T) \leq rank(A)$. Through the exact same arguments, it holds that $rank(A) \leq rank(A^T)$. Hence, the theorem follows.

Fact:

1. $0 \leq rank(A) \leq \min\{m, n\}$
2. $rank(A)$ is equal to
   (a) maximum number of linearly independent columns of A
   (b) maximum number of linearly independent rows of A

Proof: Notice that $\mathcal{R}(A)$ is the span of the columns of A.
1) If $n \geq m$, $rank(A) \leq m$ since $\mathcal{R}(A) \subset \mathbf{R}^m$.
If $n < m$, $rank(A) \leq n$ since $rank(A) + nullity(A) = n$.
2) Obvious from the above theorem.

Ex: Let $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & -2 & 4 \end{bmatrix}$. Then $rank(A) = rank(A^T) = 1$.

Elementary row operation (e.r.o.):

1. interchange two rows

2. multiply a row by a nonsero scalar

3. replace a row with the sum of itself and another row

Elementary column operations (e.c.o.) are defined similarly.

Fact: An e.r.o. (e.c.o) corresponds to premultiplying (postmultiplying) $\boldsymbol{A}$ by a left (right) elementary matrix.

Ex.: Let $\boldsymbol{A}$ be a 3×3 matrix.

1) interchange the 1st and 3rd rows

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \end{bmatrix}.$$

2) multiply the 2nd row by 2

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 2a_{21} & 2a_{22} & 2a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

3) replace the 3rd row with the sum of the 2nd and the 3rd rows

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 2a_{21} & 2a_{22} & 2a_{23} \\ a_{31} + a_{21} & a_{32} + a_{22} & a_{33} + a_{23} \end{bmatrix}.$$

Fact: An elementary operation doesn't change the rank of the matrix.

Proof: 1), 2) Obvious.

3) Let $v_1, \cdots, v_n$ be linearly independent. Let $\hat{v}_n = v_n + v_i$ for $i \neq n$. Then

$$a_1 v_1 + \cdots + a_{n-1} v_{n-1} + a_n \hat{v}_n = 0$$

$$\Downarrow$$

$$a_1 v_1 + \cdots + a_{i-1} v_{i-1} + (a_i + a_n) v_i + a_{i+1} v_{i+1} + \cdots + a_n v_n = 0$$

$$\Downarrow$$

$$a_1 = \cdots = a_n = 0.$$

2.5 System of Linear Equations

Consider a system of linear equations:

$$Ax = b \qquad (*)$$

where A is an $m \times n$ matrix.

Fundamental Theorem:

1. (*) has solutions iff A and $\tilde{A} = [A\, b]$ have the same rank.
2. If the rank is n, then (*) has a unique solution.
3. If the rank is less than n, then (*) has infinitely many solutions that can be parametrized in terms of $n - r$ unknowns.

Proof: 1) ($\Leftarrow$) b is linearly dependent on the columns of A. In other words, b is in the span of columns of A that is equal to $\mathcal{R}(A)$. Hence, (*) has solutions.

($\Rightarrow$) b is the linear combination of the columns of A and, thus, is linearly dependent on the columns of A. Hence A and $\tilde{A}$ has the same rank.

2) The n columns of A are linearly independent. If x_1 and x_2 are two different solutions. Then

$$0 = Ax_1 - Ax_2 = A(x_1 - x_2)$$

and thus n columns of A are linearly dependent since $x_1 = x_2 \neq 0$. This is a contradiction.

3) Let $rank(A) = r$. Let $c_1, \cdots, c_n$ be the columns of A. WLOG assume the first r columns of A are linearly independent. Then

$$c_i = d_{i1}c_1 + \cdots + d_{i,r}c_r, \quad i = r+1, \cdots, n.$$

Let $[\hat{x}_1 \ \cdots \ \hat{x}_r]^T$ be a unique solution of $[c_1 \ \cdots \ c_r][\hat{x}_1 \ \cdots \ \hat{x}_r]^T = b$. Then $[\hat{x}_1 \ \cdots \ \hat{x}_r \ 0 \ \cdots \ 0]^T$ is a solution of (*) and thus

$$b = \hat{x}_1 c_1 + \cdots + \hat{x}_r c_r$$

$$= \hat{x}_1 c_1 + \cdots + \hat{x}_r c_r - p_{r+1}c_{r+1} - \cdots - p_n c_n + p_{r+1}c_{r+1} + \cdots + p_n c_n$$

$$= (\hat{x}_1 - d_{r+1,1}p_{r+1} - \cdots - d_{n1}p_n)c_1 + \cdots + (\hat{x}_r - d_{r+1,r}p_{r+1} - \cdots - d_{nr}p_n)c_r$$

$$+p_{r+1}c_{r+1} + \cdots + p_n c_n$$

for any $p_i \in \mathbf{R}$. Hence, (*) has infinitely many solutions:

$$x = \begin{bmatrix} \hat{x}_1 - d_{r+1,1}p_{r+1} - \cdots - d_{n1}p_n \\ \vdots \\ \hat{x}_r - d_{r+1,r}p_{r+1} - \cdots - d_{nr}p_n \\ p_{r+1} \\ \vdots \\ p_n \end{bmatrix}$$

that are parametrized in terms of $n - r$ unknowns p_i.

Consider a homogeneous system of linear equations:

$$\boldsymbol{A}x = 0 \qquad (**)$$

Theorem:

1. (**) always has the trivial solution $x = 0$.
2. Nontrivial solutions exist iff $r := rank\boldsymbol{A} < n$.
3. If $r < n$, the set of all solutions to (**) is the null space of $\boldsymbol{A}$ that is an $n - rank(\boldsymbol{A})$ dimensional subspace.

Proof: 1) Obvious.

2) Obvious from Fundamental Theorem 2) and 3).

3) Obvious from the definitions of null space and nullity.

Construction of null space:

Due to homogeneity, we can assume WLOG $\hat{x}_1 = \cdots = \hat{x}_r = 0$ in the proof of Fundamental Theorem 3). Let

$$y_j = [-d_{j1} \ \cdots \ -d_{jr} \ 0 \ \cdots \ \underbrace{1}_{j\text{th}} \ \cdots \ 0]^T, \quad j = r+1, \cdots, n.$$

Then it holds that any solution x can be written as

$$x = \begin{bmatrix} -d_{r+1,1}p_{r+1} - \cdots - d_{n1}p_n \\ \vdots \\ -d_{r+1,r}p_{r+1} - \cdots - d_{nr}p_n \\ p_{r+1} \\ \vdots \\ p_n \end{bmatrix} = p_{r+1}y_{r+1} + \cdots + p_n y_n.$$

Hence,

$$\mathcal{N}(A) = Span\{y_{r+1}, \cdots, y_n\}.$$

Theorem: Suppose (*) has solutions. Let x_0 be any solution of (*). Then x is a solution of (*) iff

$$x = x_0 + x_h, \quad x_h \in \mathcal{N}(A).$$

Proof: ($\Leftarrow$) Obvious.

($\Rightarrow$) Notice that

$$A(x - x_0) = Ax - Ax_0 = b - b = 0.$$

Hence, $x_h := x - x_0 \in \mathcal{N}(A)$.

2.6 Determinant of Square Matrices

Definitions:

- $$det(A) = \sum_{\phi \in P_n} sign\phi \prod_{i=1}^{n} a_{i\phi(i)}$$

  where P_n is the set of all permutations of $\{1, \cdots, n\}$ and $sign\phi = \pm 1$ depending on whether ϕ is an even or odd permutation.

- minor of a_{ij}: $det(M_{ij})$ where M_{ij} is the matrix obtained deleting ith row and jth column.

- leading (principal) minors: $det(M_{ii})$

- cofactor of a_{ij}:

  $$C_{ij} = (-1)^{i+j} det(M_{ij})$$

- $$det(A) = \sum_{i=1}^{n} a_{ij} C_{ij} = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} det(M_{ij}) \quad \text{column expansion}$$

  $$det(A) = \sum_{j=1}^{n} a_{ij} C_{ij} = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} det(M_{ij}) \quad \text{row expansion}$$

Second order determinant:

$$D = det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Third order determinant:

$$D = det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

Properties of Determinant:

1. $det(A) = det(A^T)$

   Proof for 2×2 case:

$$det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} = \begin{vmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{vmatrix} = det(A^T).$$

2. $det(aA) = a^n det(A)$

   Proof: Obvious from the difinition of $det(A)$.

3. $det(AB) = det(A)det(B)$.

   Proof for 2×2 case:

$$det(AB) = \begin{vmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{vmatrix}$$

$$= (a_{11}b_{11} + a_{12}b_{21})(a_{21}b_{12} + a_{22}b_{22}) - (a_{11}b_{12} + a_{12}b_{22})(a_{21}b_{11} + a_{22}b_{21})$$

$$= a_{11}a_{21}\underbrace{(b_{11}b_{12} - b_{12}b_{11})}_{=0} + a_{11}a_{22}(b_{11}b_{22} - b_{12}b_{21})$$

$$+ a_{12}a_{21}(b_{21}b_{12} - b_{22}b_{11}) + a_{12}a_{22}\underbrace{(b_{21}b_{22} - b_{22}b_{21})}_{=0}$$

$$= (a_{11}a_{22} - a_{12}a_{21})(b_{11}b_{22} - b_{12}b_{21})$$

$$= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} = det(\boldsymbol{A})det(\boldsymbol{B}).$$

4. If a row or column of $\boldsymbol{A}$ is zero, then $det(\boldsymbol{A}) = 0$.

   Proof: Obvious from the row and column expansion.

5. If a row (column) of $\boldsymbol{A}$ is a sum of two row (column) vectors, then $det(\boldsymbol{A}) = det(\boldsymbol{A}') + det(\boldsymbol{A}'')$ where $\boldsymbol{A}'$ has one of row (column) vector and $\boldsymbol{A}''$ has the other.

   Ex:
$$\boldsymbol{A} = \begin{bmatrix} a_{11} + b_{11} & a_{12} & a_{13} \\ a_{21} + b_{21} & a_{22} & a_{23} \\ a_{31} + b_{31} & a_{32} & a_{33} \end{bmatrix}.$$

   Then,
$$\boldsymbol{A}' = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \boldsymbol{A}'' = \begin{bmatrix} b_{11} & a_{12} & a_{13} \\ b_{21} & a_{22} & a_{23} \\ b_{31} & a_{32} & a_{33} \end{bmatrix}.$$

   Proof: Obvious from the row and column expansion.

   Ex:
$$det(\boldsymbol{A}) = \sum_{i=1}^{3}(-1)^{i+1}(a_{i1} + b_{i1})det(\boldsymbol{M}_{i1})$$

$$= \sum_{i=1}^{3}(-1)^{i+1}a_{i1}det(\boldsymbol{M}_{i1}) + \sum_{i=1}^{3}(-1)^{i+1}b_{i1}det(\boldsymbol{M}_{i1}) = det(\boldsymbol{A}') + det(\boldsymbol{A}'').$$

6. Let $\boldsymbol{A}$ be a triangular matrix. Then $det(\boldsymbol{A}) = \Pi_{i=1}^{n} a_{ii}$.

   Proof for 3×3 case:
$$\begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33}.$$

Theorem 1: If $\tilde{A}$ is obtained by interchanging two rows (columns) of A, then $det(\tilde{A}) = -det(A)$.

Proof: For $n = 2$, the theorem is clearly true. Suppose $(n-1)$th order determinant has the property. Expand $det(A)$ and $det(\tilde{A})$ by a row that is not one of those interchanged, call it the ith row. Then

$$det(A) = \sum_{j=1}^{n}(-1)^{i+j}a_{ij}det(M_{ij}), \quad det(\tilde{A}) = \sum_{j=1}^{n}(-1)^{i+j}a_{ij}det(\tilde{M}_{ij})$$

where $\tilde{M}_{ij}$ is obtained from the minor M_{ij} by interchanging two rows. Then $det(\tilde{M}_{ij}) = -det(M_{ij})$ and thus $det(\tilde{A}) = -det(A)$.

Theorem 2: If two rows (columns) of A are proportional, then $det(A) = 0$.

Proof: Let $a_{ik} = ca_{jk}$ for $k = 1, \cdots, n$. If $c = 0$, then $det(A) = 0$. Let $c \neq 0$. Then

$$det(A) = c\ det(\tilde{A})$$

where $\tilde{A}$ is obtained by dividing the ith row of A by c. Interchanging the ith and jth rows of $\tilde{A}$,

$$det(\tilde{A}) = -det(\tilde{A}).$$

Hence, $det(\tilde{A}) = 0$ and, thus, $det(A) = c\ det(\tilde{A}) = 0$.

Theorem 3: The value of a determinant is left unchanged if a row (column) is replaced with the row (column) subtracted by a constant multiple of any other row (column).

Proof: Obvious from Property 5 and Theorem 2.

Theorem 4: The rank of an $n \times n$ matrix A is n iff $det(A) \neq 0$.

Proof: $(\Leftarrow)$ Suppose the contrary; $rank(A) < n$. Then the columns of A are linearly dependent. WLOG assume the first column of A is linearly dependent on the others. Let $A = [c_1 \ \cdots \ c_n]$. Then $c_1 = \sum_{i=2}^{n} a_i c_i$. Define

$$\tilde{A} = [\underbrace{c_1 - \sum_{i=2}^{n} a_i c_i}_{=0} \ c_2 \cdots c_n].$$

Then $det(A) = det(\tilde{A}) = 0$.

$(\Rightarrow)$ (2×2 case) Suppose the contrary. If a row is *zero*, then the rows of A are linearly dependent and, thus, $rank(A) < 2$. Otherwise, a_{ij}'s are all nonzero and

$$0 = det(A) = a_{11}a_{22} - a_{12}a_{21}.$$

$$\Downarrow$$
$$\frac{a_{11}}{a_{12}} = \frac{a_{21}}{a_{22}} = c.$$
$$\Downarrow$$
$$\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} = c \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}.$$

Hence, the columns of $\boldsymbol{A}$ are linearly dependent and, thus, $rank(\boldsymbol{A}) < 2$.

Lemma: Let $\boldsymbol{A}$ be a $m \times n$ matrix where $m \geq n$. Then $rank(\boldsymbol{A}) < n$ iff the determinant of every $n \times n$ square submatrix is zero.

Proof: ($\Rightarrow$) Similar to Theorem 4, replace a linearly dependent row with the zero row vector that is equal to the first row subtracted by suitable constant multiples of other rows.

($\Leftarrow$) Suppose the contrary. Let $\tilde{\boldsymbol{A}}$ be the $n \times n$ matrix containing n linearly independent rows of $\boldsymbol{A}$. Then $rank(\tilde{\boldsymbol{A}}) = n$ and thus $det(\tilde{\boldsymbol{A}}) \neq 0$.

Corollary: Let $\boldsymbol{A}$ be a $m \times n$ matrix where $m \geq n$. Then $rank(\boldsymbol{A}) = n$ iff there exists an $n \times n$ square submatrix whose determinant is not zero.

Theorem 5: An $m \times n$ matrix $\boldsymbol{A}$ has rank r iff $\boldsymbol{A}$ has an $r \times r$ submatrix with nonzero determinant whereas the determinant of every square submatrix with $r+1$ or more rows is zero.

Proof: ($\Leftarrow$) From Lemma, any $r+1$ columns are linearly dependent and there exists r columns that are linearly independent. Hence, the rank is r.

($\Rightarrow$) Obvious from Lemma since there exists r linearly independent columns and any set of $r+1$ columns are linearly dependent.

2.7 Inverse of Square Matrices

Def.: Inverse of an $n \times n$ matrix $\boldsymbol{A}$ is an $n \times n$ matrix such that

$$\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I}.$$

Def.: If $\boldsymbol{A}$ has an inverse, it is called nonsingular. Otherwise, it is called singular.

Fact: The inverse of a nonsingular matrix $\boldsymbol{A}$ is unique.

Proof: Let $\boldsymbol{B}$ and $\boldsymbol{C}$ be the inverses of $\boldsymbol{A}$. Then $\boldsymbol{A}\boldsymbol{B} = \boldsymbol{I}$ and $\boldsymbol{C}\boldsymbol{A} = \boldsymbol{I}$ and thus

$$\boldsymbol{B} = \boldsymbol{I}\boldsymbol{B} = (\boldsymbol{C}\boldsymbol{A})\boldsymbol{B} = \boldsymbol{C}(\boldsymbol{A}\boldsymbol{B}) = \boldsymbol{C}\boldsymbol{I} = \boldsymbol{C}.$$

Theorem: TFAE

1. An $n \times n$ matrix $\boldsymbol{A}$ has its inverse. In other words, $\boldsymbol{A}$ is nonsingular.
2. $rank(\boldsymbol{A}) = n$.
3. $det\boldsymbol{A} \neq 0$.
4. The columns of $\boldsymbol{A}$ are linearly independent.

Proof: $(1 \Rightarrow 2)$ For any b, $\boldsymbol{A}x = b$ has the solution $x = \boldsymbol{A}^{-1}b$. Hence, $rank(\boldsymbol{A}) = n$.

$(1 \Leftarrow 2)$ For any b, $\boldsymbol{A}x = b$ has the solution $x(b)$. Let x_1, x_2 be the solutions associated with b_1, b_2, respectively. Then it is clear that $a_1x_1 + a_2x_2$ is the solution associated with $a_1b_1 + a_2b_2$. Hence $x(b)$ is a linear operator. By the matrix representation theorem, $x(b) = \boldsymbol{B}b$. Then for any b,

$$b = \boldsymbol{A}x(b) = \boldsymbol{A}(\boldsymbol{B}b) = (\boldsymbol{AB})b,$$

$$x(b) = \boldsymbol{B}b = \boldsymbol{B}(\boldsymbol{A}x(b)) = (\boldsymbol{BA})x(b).$$

Hence $\boldsymbol{B} = \boldsymbol{A}^{-1}$.

$(2 \Leftrightarrow 3)$ Obvious from Theorem 4 in the previous section.

$(2 \Leftrightarrow 4)$ proven in the previous rank and nullity section.

Some Useful Inverse Formulas:

- (Cramer's rule):

  The adjoint of $\boldsymbol{A}$ is defined by $adj\boldsymbol{A} = [C_{ji}]$. Then

  $$adj(\boldsymbol{A}) \cdot \boldsymbol{A} = \boldsymbol{A} \cdot adj(\boldsymbol{A}) = det(\boldsymbol{A}) \cdot \boldsymbol{I}.$$

  If $\boldsymbol{A}$ has an inverse,

  $$\boldsymbol{A}^{-1} = \frac{adj(\boldsymbol{A})}{det(\boldsymbol{A})}.$$

  2×2 example:

  $$\boldsymbol{A}^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}.$$

- If $A = diag\{d_1, \cdots, d_n\}$, then $A^{-1} = diag\{\frac{1}{d_1}, \cdots, \frac{1}{d_n}\}$.
- $(A^{-1})^{-1} = A$.
- $(AB)^{-1} = B^{-1}A^{-1}$.
- Partitioned inverse for $\mathcal{R}^{n \times n}$

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^{-1} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

where

$$B_{11} = A_{11}^{-1} + (A_{11}^{-1}A_{12})Z^{-1}(A_{21}A_{11}^{-1})$$
$$B_{12} = -(A_{11}^{-1}A_{12})Z^{-1}$$
$$B_{21} = -Z^{-1}(A_{21}A_{11}^{-1})$$
$$B_{22} = -Z^{-1}$$
$$Z = A_{22} - A_{21}(A_{11}^{-1}A_{12}).$$

Theorem: Let A, B, C be $n \times n$ matrices.

1. If $rank(A) = n$ and $AB = AC$, then $B = C$.
2. If $rank(A) = n$, then $AB = 0$ implies $B = 0$. Hence if $AB = 0$ but $A \neq 0$ as well as $B \neq 0$, then $rank(A) < n$ and $rank(B) < n$.
3. If A is singular, so are AB and BA.

Proof: 1) Premultiply A^{-1}.
2) Obvious from 1).
3) $det(AB) = det(BA) = det(A)det(B) = 0$.

2.8 Change of Basis: Coordinate Transform

Let $\{u_k\}_{k=1}^n$ and $\{\tilde{u}_i\}_{i=1}^n$ be two bases for $\mathbf{R}^n$ and $\{v_k\}_{k=1}^m$ and $\{\tilde{v}_i\}_{i=1}^m$ two bases for $\mathbf{R}^m$. Then

$$\tilde{u}_i = \sum_{k=1}^n p_{ki}u_k.$$

$$\Downarrow$$

$$\sum_{k=1}^{n} \xi_k u_k = x = \sum_{i=1}^{n} \tilde{\xi}_i \tilde{u}_i = \sum_{i,k=1}^{n} p_{ki} \tilde{\xi}_i u_k.$$

$$\Downarrow$$

$$\xi_k = \sum_{i=1}^{n} p_{ki} \tilde{\xi}_i.$$

$$\Downarrow$$

$$\begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix} = \begin{bmatrix} p_{11} & \cdots & p_{1n} \\ \vdots & \ddots & \vdots \\ p_{n1} & \cdots & p_{nn} \end{bmatrix} \begin{bmatrix} \tilde{\xi}_1 \\ \vdots \\ \tilde{\xi}_n \end{bmatrix}.$$

$$\Downarrow$$

$$\xi = \boldsymbol{P}\tilde{\xi}.$$

Notice that the ith column of $\boldsymbol{P}$ is the representation of $\tilde{u}_i$ w.r.t $\{u_j\}$.

Similarly,

$$\tilde{\eta} = \boldsymbol{Q}\eta.$$

Notice that the ith column of $\boldsymbol{Q}$ is the representation of v_i w.r.t $\{\tilde{v}_j\}$.

Let $y = \boldsymbol{A}x \Rightarrow \eta = \boldsymbol{A}\xi \Rightarrow$

$$\tilde{\eta} = \boldsymbol{QA}\xi = \boldsymbol{QAP}\tilde{\xi}.$$

$$\Downarrow$$

the representation of linear operator w.r.t. $\{\tilde{u}_i\}$ and $\{\tilde{v}_i\}$ is

$$\tilde{\boldsymbol{A}} = \boldsymbol{QAP}.$$

Special Case: $m = n$ and use same basis for both domain and range. Then

$$\boldsymbol{PQ} = \boldsymbol{I} \;\Rightarrow\; \boldsymbol{Q} = \boldsymbol{P}^{-1} \;\Rightarrow\; \tilde{\boldsymbol{A}} = \boldsymbol{P}^{-1}\boldsymbol{AP}.$$

Such transformation from $\boldsymbol{A}$ to $\tilde{\boldsymbol{A}}$ is called similarity transformation.

2.9 Eigenvalues and Eigendecomposition

Def: $\lambda \in \mathbf{C}$ is called an eigenvalue of a square matrix $\boldsymbol{A}$ if $\exists$ right (left) eigenvector $x(y) \neq 0$ such that $\boldsymbol{A}x = \lambda x$ ($y^* \boldsymbol{A} = \lambda y^*$).

Fact: If v is an eigenvector associated with λ, so is αv with $\alpha \neq 0$.

Fact: λ is an eigenvalue of $\boldsymbol{A}$ iff it is a solution of the characteristic polynomial

$$\chi_A(\lambda) = det(\boldsymbol{A} - \lambda \boldsymbol{I}) = 0.$$

Fact: The eigenvector v associated with an eigenvalue λ is a nonzero vector in the null space of $\lambda \boldsymbol{I} - \boldsymbol{A}$. Hence, the number of linearly independent eigenvectors is equal to $nullity(\boldsymbol{A} - \lambda \boldsymbol{I})$.

Theorem: Let $\lambda_1, \cdots, \lambda_n$ be the distinct eigenvalues of $\boldsymbol{A}$ and v_i be an eigenvalue associated with λ_i. Then $\{v_i\}_{i=1}^n$ is linearly independent.

Proof: Suppose the contrary. $\exists \alpha_i$'s (not all zero) such that

$$\alpha_1 v_1 + \cdots + \alpha_n v_n = 0.$$

WLOG, we assume $\alpha_1 \neq 0$. Then

$$(\boldsymbol{A} - \lambda_2 \boldsymbol{I}) \cdots (\boldsymbol{A} - \lambda_n \boldsymbol{I}) \left(\sum_{i=1}^{n} \alpha_i v_i \right) = 0.$$

Notice that

$$(\boldsymbol{A} - \lambda_j \boldsymbol{I}) v_i = (\lambda_i - \lambda_j) v_i \quad \text{if } j \neq i$$

and

$$(\boldsymbol{A} - \lambda_i \boldsymbol{I}) v_i = 0.$$

Hence,

$$\alpha_1 (\lambda_1 - \lambda_2) \cdots (\lambda_1 - \lambda_n) v_1 = 0.$$

Since λ_i's are distinct, this implies $\alpha_1 = 0$ (contradiction!).

Def.: A square matrix is simple if it has n linearly independent eigenvectors.

Corollary: If eigenvalues of $\boldsymbol{A}$ are all distinct, $\boldsymbol{A}$ is simple.

Fact: Suppose $\boldsymbol{A}$ has some multiple eigenvalues. Then $\boldsymbol{A}$ is simple iff $nullity(\boldsymbol{A} - \lambda \boldsymbol{I}) = p$ for any eigenvalue λ with multiplicity p.

Ex: Suppose

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then $\lambda = 1$ is a double eigenvalue and $nullity(A - \lambda) = nullity(0) = 2$. Hence, we can find two linearly independent eigenvectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. However suppose

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Then $\lambda = 1$ is again a double eigenvalue but

$$nullity(A - \lambda) = nullity\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = 1.$$

Hence, we can find only one linearly independent eigenvector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Let A be simple. Note that

$$AV = V\Lambda$$

where

$$V = [v_1 \ \cdots \ v_n], \qquad \Lambda = diag\{\lambda_1, \cdots, \lambda_n\}.$$

Since V is nonsingular, we have

$$V^{-1}AV = \Lambda.$$

Note that Λ is the representation of A in terms of its eigenvectors.

Fact: Eigenvalues are invariant under similarity transform.

Proof: Suppose $Av = \lambda v$. Let $\tilde{A} = P^{-1}AP$ and $\tilde{v} = P^{-1}v$. Then

$$\tilde{A}\tilde{v} = P^{-1}APP^{-1}v = \lambda P^{-1}v = \lambda\tilde{v}.$$

2.10 Hermitian and Unitary Matrices

Def.: $A \in \mathbf{C}^{n\times n}(\mathbf{R}^{n\times n})$ is Hermitian (symmetric) if $A^* = A (A^T = A)$.

Theorem: Let A be Hermitian.

1. x^*Ax is real.

2. eigenvalues of A are all real.

3. n eigenvectors exist and are all orthogonal.

Proof: 1) $(x^*Ax)^* = x^*A^*x = x^*Ax$.

2) Let λ be an eigenvalue and v be the corresponding eigenvector. Then $v^*Av = \lambda v^*v$. Note that LHS is real and v^*v is real and > 0.

3) (Proof of orthogonality) For multiple eigenvalues, we can always choose mutually orthogonal eigenvectors. Suppose $Au = \lambda u$ and $Av = \mu v$ with $\lambda \neq \mu$. Note that $u^*A = \lambda u^*$. Hence

$$u^*Av = \lambda u^*v \quad \text{and} \quad u^*Av = \mu u^*v$$

$\Rightarrow \lambda u^*v = \mu u^*v \Rightarrow u^*v = 0$.

Def.: A Hermitian matrix A is positive semidefinite (PSD) if $x^*Ax \geq 0$ for all x.

Def.: A Hermitian matrix A is positive definite (PD) if $x^*Ax > 0$ for all $x \neq 0$.

Theorem: Let A be a Hermitian matrix. Then TFAE

1. A is PSD (PD)

2. all its eigenvalues are nonnegative (positive)

Proof: $(1 \Rightarrow 2)$ Let λ_i be an eigenvalue and v_i be the corresponding unit eigenvector. Then

$$Av_i = \lambda_i v_i \quad \Rightarrow \quad 0 \leq (<) v_i^* A v_i = \lambda_i v_i^* v_i = \lambda_i.$$

$(2 \Rightarrow 1)$ $\{v_i\}$ orthonormal eigenvectors

$$Ax = A(a_1 v_1 + \cdots + a_n v_n) = a_1 A v_1 + \cdots + a_n A v_n = a_1 \lambda_1 v_1 + \cdots + a_n \lambda_n v_n$$

$$\Downarrow$$

$$x^*Ax = (a_1 v_1^* + \cdots + a_n v_n^*)(a_1 \lambda_1 v_1 + \cdots + a_n \lambda_n v_n) = a_1^2 \lambda_1 + \cdots + a_n^2 \lambda_n \geq (>) 0.$$

Def.: $A \in \mathbf{C}^{n \times n}(\mathbf{R}^{n \times n})$ is unitary (orthogonal) if $A^* = A^{-1}$.

Theorem: Let A be Unitary. Then the eigenvalues of A have absolute value 1.

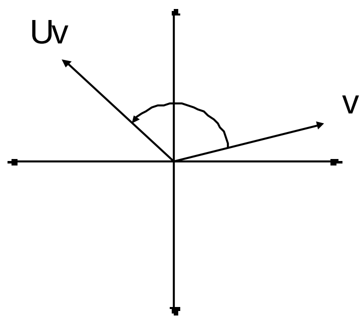
Proof: Let λ_i be an eigenvalue and v_i be the corresponding unit eigenvector. Then

$$1 = |v_i|^2 = v^* v = v^* A^* A v = v^* \bar{\lambda} \lambda v = |\lambda|^2 |v|^2 = |\lambda|^2.$$

Theorem: If A is unitary, $x_1^* x_2 = x_1^* A^* A x_2 = y_1^* y_2$ where $y_1 = Ax_1$ and $y_2 = Ax_2$.

Remark 1: From the above theorem, the inner product is preserved under unitary transformation.

Remark 2: From Remark 1, the norm is preserved under unitary transformation. Hence, unitary matrix rotates a vector without change of size:



U: unitary matrix

Theorem: A square matrix is unitary iff its columns (or rows) are orthonormal each other.

Proof: Obvious from $A^* A = I$.

Theorem: The determinant of a unitary matrix has absolute value 1.

Proof:

$$1 = det(I) = det(A^* A) = det(A^*) det(A) = \overline{det(A)} det(A) = |det(A)|^2.$$

Let A be a Hermitian matrix and $\{v_i\}$ be the set of n orthonormal eigenvectors. Let U be the unitary matrix whose ith column is v_i. Then the representation of A in terms of its orthonormal eigenvectors is

$$\Lambda = U^* A U.$$

2.11 Singular Values and Singular Value Decomposition

Def.: Singular values of an $m \times n$ matrix A are the square roots of $\min\{m, n\}$ eigenvalues of $A^* A$.

$$\sigma(A) = \sqrt{\lambda(A^* A)}.$$

Def.: Right singular vectors of a matrix A are the eigenvectors of $A^* A$:

$$\sigma(A)^2 v - A^* A v = 0.$$

Left singular vectors of a matrix A are the eigenvectors of AA^*:

$$\sigma(A)^2 u - AA^* u = 0.$$

Remark: Since $A^* A$ is a PSD Hermitian matrix, singular values are nonnegative reals and n singular vectors can be chosen so that they are orthonormal.

Theorem (Singular Value Decomposition): Let $A \in \mathbf{R}^{m \times n}$. Suppose σ_i be singular values of A such that

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_p \geq 0, \quad p = \min\{m, n\}.$$

Let

$$U = [u_1, u_2, \cdots, u_m] \in \mathbf{R}^{m \times m}, \qquad V = [v_1, v_2, \cdots, v_n] \in \mathbf{R}^{n \times n},$$

where u_i, v_j denote left and right orthonormal singular vectors of A, respectively. Let

$$\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix}$$

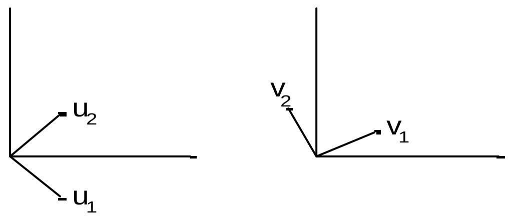
where

$$\Sigma_1 = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \sigma_p \end{bmatrix}.$$

Then

$$A = U\Sigma V^* = \sum_{i=1}^{p} \sigma_i(A) u_i v_i^*.$$

Proof: Consider $y = Ax$. Then Σ is simply the representation of A when x and y are represented in the coordinate systems consisting of right and left singular vectors, respectively.



Remark:

$$Av_1 = \sigma_1 u_1$$

$$Av_p = \sigma_p u_p$$

$$\Downarrow$$

v_1 (v_p): highest (lowest) gain input direction.

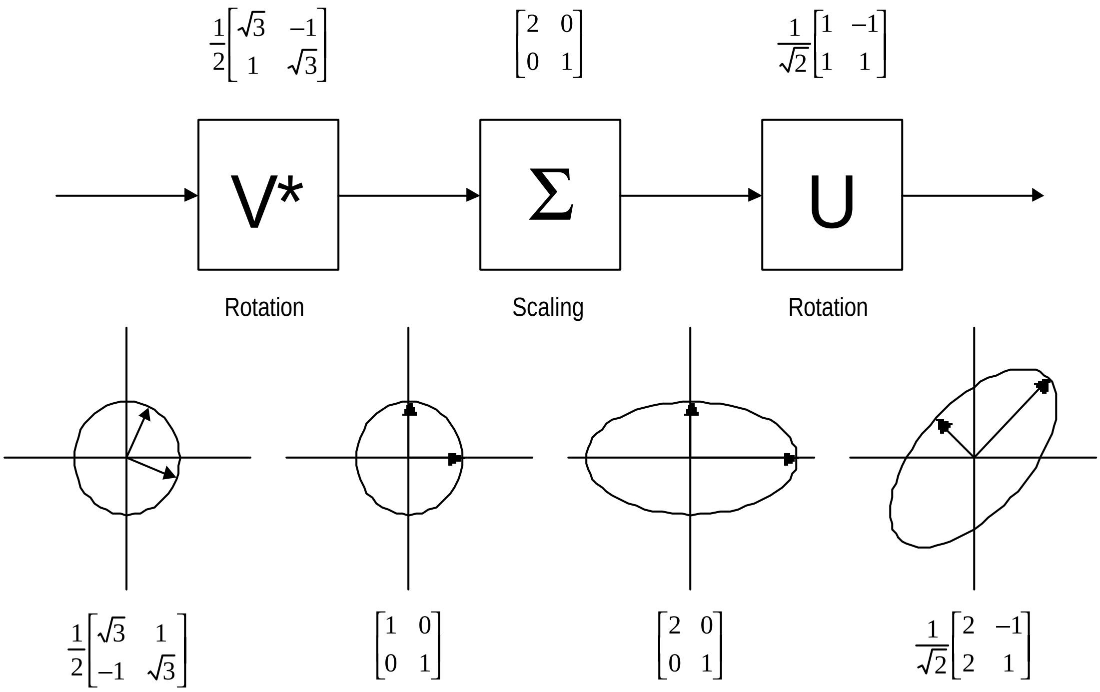
u_1 (u_p): highest (lowest) gain observing direction.

Example:

$$A = \begin{bmatrix} 0.8712 & -1.3195 \\ 1.5783 & -0.0947 \end{bmatrix}$$

$$\Downarrow$$

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad V = \frac{1}{2} \begin{bmatrix} \sqrt{3} & 1 \\ -1 & \sqrt{3} \end{bmatrix}$$



2.12 Matrix Norms

Norms for $\mathbf{R}^{m\times n}$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \in \mathbf{R}^{m\times n}$$

p norms:

$$|||A|||_p = (\sum_{i,j} |a_{i,j}|^p)^{\frac{1}{p}} \quad 1 \le p < \infty$$

$$|||A|||_\infty = \max_{i,j} |a_{i,j}|$$

$||| \cdot |||_2$ is called the Frobenius norm.

What is the difference between $\mathbf{R}^{m \times n}$ and $\mathbf{R}^{mn}$?

A matrix in $\mathbf{R}^{m \times n}$ defines a linear operator from $\mathbf{R}^n$ to $\mathbf{R}^m$; $y = Ax$.

induced (or operator) p norms:

$$||A||_p = \sup_{x \ne 0} \frac{||Ax||_p}{||x||_p} = \max_{||x||=1} ||Ax||_p \quad 1 \le p \le \infty$$

$$\Downarrow$$

$$||y||_p = ||Ax||_p \le ||A||_p ||x||_p \quad \forall x \in \mathbf{R}^n.$$

Ex:

$p = 1$:

$$||A||_1 = \max_j \sum_{i=1}^m |a_{i,j}|.$$

$p = 2$:

$$||A||_2 = \sigma_{max}(A) := [\lambda_{max}(A^T A)]^{\frac{1}{2}}$$

where σ_{max} is the maximum singular value of A.

$p = \infty$:

$$||A||_\infty = \max_i \sum_{j=1}^n |a_{i,j}|.$$

Chapter 3

Computational Linear Algebra

3.1 Gauss-Jordan Elimination

Question: How can we find the solution of the system of linear equations:

$$Ax = b?$$

A unique solution exists iff A is nonsingular. Hence, we assume A is nonsingular throughout this chapter.

If A is nonsingular, it is clear that the unique solution is $x = A^{-1}b$. However, the solution can be found without computing A^{-1} using more efficient method. Hence, if what you want is only the solution but not A^{-1}, do not try to find A^{-1}.

A standard technique to find the solution of the above problem is the so called Gaussian elimination.

We illustrate the Gaussian elimination using a 3×3 system:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

Main idea is to transform A into upper triangular matrix and solve the resulting problem by backward substitution.

Forward Elimination:

- Rearrange the rows so that a_{11} nonzero (This is always possible if A is nonsingular).

- 2nd equation - $\frac{a_{21}}{a_{11}}\times$ first equation
  3rd equation - $\frac{a_{31}}{a_{11}}\times$ first equation

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} - \frac{a_{21}}{a_{11}}a_{12} & a_{23} - \frac{a_{21}}{a_{11}}a_{13} \\ 0 & a_{32} - \frac{a_{31}}{a_{11}}a_{12} & a_{33} - \frac{a_{31}}{a_{11}}a_{13} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 - \frac{a_{21}}{a_{11}}b_1 \\ b_3 - \frac{a_{31}}{a_{11}}b_1 \end{bmatrix}.$$

$$\Downarrow$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} \\ 0 & a_{32}^{(1)} & a_{33}^{(1)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2^{(1)} \\ b_3^{(1)} \end{bmatrix}$$

  where

$$a_{22}^{(1)} = a_{22} - \frac{a_{21}}{a_{11}}a_{12}$$
$$a_{23}^{(1)} = a_{23} - \frac{a_{21}}{a_{11}}a_{13}$$
$$a_{32}^{(1)} = a_{32} - \frac{a_{31}}{a_{11}}a_{12}$$
$$a_{33}^{(1)} = a_{33} - \frac{a_{31}}{a_{11}}a_{13}$$
$$b_2^{(1)} = b_2 - \frac{a_{21}}{a_{11}}b_1$$
$$b_3^{(1)} = b_3 - \frac{a_{31}}{a_{11}}b_1.$$

- Rearrange the rows so that $a_{22}^{(1)}$ nonzero (This is always possible if $\boldsymbol{A}$ is nonsingular).

- 3rd equation - $\frac{a_{32}^{(1)}}{a_{22}^{(1)}}\times$ 2nd equation

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} \\ 0 & 0 & a_{33}^{(1)} - \frac{a_{32}^{(1)}}{a_{22}^{(1)}}a_{23}^{(1)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2^{(1)} \\ b_3^{(1)} - \frac{a_{32}^{(1)}}{a_{22}^{(1)}}b_2^{(1)} \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} \\ 0 & 0 & a_{33}^{(2)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2^{(1)} \\ b_3^{(2)} \end{bmatrix}$$

where

$$a_{33}^{(2)} = a_{33}^{(1)} - \frac{a_{32}^{(1)}}{a_{22}^{(1)}} a_{23}^{(1)}$$

$$b_3^{(2)} = b_3^{(1)} - \frac{a_{32}^{(1)}}{a_{22}^{(1)}} b_2^{(1)}.$$

Backward Substitution:

$$x_3 = \frac{b_3^{(2)}}{a_{33}^{(2)}}$$

$$x_2 = \frac{b_2^{(1)} - a_{23}^{(1)} x_3}{a_{22}^{(1)}}$$

$$x_1 = \frac{b_1 - a_{13}x_3 - a_{12}x_2}{a_{11}}$$

Remark: Notice that the elimination step can be done in the matrix $[\boldsymbol{A}\, b]$.

Indeed $\boldsymbol{A}^{-1}$ can also be found using elimination method called Gauss-Jordan elimination that consists of forward Gaussian elimination and backward Jordan elimination. Starting from

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 1 \end{bmatrix},$$

apply forward and backward elimnation to get

$$\begin{bmatrix} 1 & 0 & 0 & \alpha_{11} & \alpha_{12} & \alpha_{13} \\ 0 & 1 & 0 & \alpha_{21} & \alpha_{22} & \alpha_{23} \\ 0 & 0 & 1 & \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix}.$$

Then

$$\boldsymbol{A}^{-1} = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix}.$$

Notice that finding A^{-1} is computationally more involved than finding the solution of the system.

When we perform the Gaussian elimination in computer, the small diagonal pivot element becomes problematic due to truncation error. Consider

$$\begin{bmatrix} 0.0001 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Assume we only represent 3 significant digit. Then the solution by Cramer's rule is $x_1 = 1.00 \times 10^0$ and $x_2 = 1.00 \times 10^0$. Now apply Gaussian elimination. Then

$$a_{22}^{(1)} = a_{22} - \frac{a_{21}}{a_{11}} a_{12} = 1.00 \times 10^0 - \frac{1.00 \times 10^0}{1.00 \times 10^{-4}} 1.00 \times 10^0 = -1.00 \times 10^4$$

$$b_2^{(1)} = b_2 - \frac{a_{21}}{a_{11}} b_1 = 2.00 \times 10^0 - \frac{1.00 \times 10^0}{1.00 \times 10^{-4}} 1.00 \times 10^0 = -1.00 \times 10^4$$

Hence,

$$x_2 = \frac{b_2^{(1)}}{a_{22}^{(1)}} = \frac{-1.00 \times 10^4}{-1.00 \times 10^4} = 1.00 \times 10^0$$

$$x_1 = \frac{b_1 - a_{12} x_2}{a_{11}} = \frac{1.00 \times 10^0 - 1.00 \times 10^0}{1.00 \times 10^{-4}} = 0.0$$

Hence, the result is wrong! Notice that the problem is caused by the small pivot element a_{11}. Hence, a possible resolution of this problem is to rearrange the matrix so that the pivot elements are always large. Such technique is called pivoting.

Column pivoting: pivoting by interchanging rows

Row pivoting: pivoting by interchanging columns.

Full pivoting: pivoting by interchanging both columns and rows.

The column pivoting is the easiest since it does not scramble the variables. Notice that in column pivoting case, we are only allowed to consider rows below the pivot element as interchange candidates. Otherwise the upper triangular structure is destroyed.

If we interchange the first and the second rows in the above example, then the correct solution is obtained using Gaussian elimination.

3.2 LU Decomposition

Suppose A can be written as

$$A = LU$$

where L is a lower triangular matrix and U is an upper triangular matrix. Then

$$Ax = (LU)x = L(Ux) = b.$$

Let $y = Ux$. Then

$$Ly = b$$

$$Ux = y.$$

Then $Ax = b$ can be solved by first solving the first equation by forward substitution and then solving the second equation by backward substitution.

Question: How can we get the LU decomposition?

Consider

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ l_{31} & l_{32} & 1 & 0 \\ l_{41} & l_{42} & l_{43} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{bmatrix}$$

$$= \begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} & l_{21}u_{14} + u_{24} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} & l_{31}u_{14} + l_{32}u_{24} + u_{34} \\ l_{41}u_{11} & l_{41}u_{12} + l_{42}u_{22} & l_{41}u_{13} + l_{42}u_{23} + l_{43}u_{33} & l_{41}u_{14} + l_{42}u_{24} + l_{43}u_{34} + u_{44} \end{bmatrix}.$$

$$\Downarrow$$

$$a_{ij} = l_{i1}u_{1j} + l_{i2}u_{2j} + \cdots + u_{jj}, \quad i = j$$

$$a_{ij} = l_{i1}u_{1j} + l_{i2}u_{2j} + \cdots + u_{ij}, \quad i < j$$

$$a_{ij} = l_{i1}u_{1j} + l_{i2}u_{2j} + \cdots + l_{ij}u_{jj}, \quad i > j$$

$$\Downarrow$$

$$u_{ij} = a_{ij} - (l_{i1}u_{1j} + l_{i2}u_{2j} + \cdots + l_{i(i-1)}u_{(i-1)j}) = a_{ij} - \sum_{k=1}^{i-1} l_{ik}u_{kj} \quad i \leq j$$

$$l_{ij} = \frac{1}{u_{jj}}(a_{ij} - \sum_{k=1}^{j-1} l_{ik}u_{kj}) \quad i > j$$

We need to start from the upper left corner and work toward the lower right.

Remark 1: In the above algorithm, the pivots are diagonal elements. Hence, we may need pivoting similar to, but different from, that in Gaussian elimination.

Remark 2: $\boldsymbol{A}^{-1}$ can also be found using LU decomposition. Let α_i be the solution to

$$\boldsymbol{A}\alpha_i = \boldsymbol{L}\boldsymbol{U}\alpha_i = e_i.$$

Then

$$\boldsymbol{A}^{-1} = [\alpha_1\ \alpha_2\ \cdots\ \alpha_n].$$

Thomas Algorithm (LU decomposition for tridiagonal matrix)

Consider

$$\begin{bmatrix} b_1 & c_1 & 0 & 0 & \cdots & 0 \\ a_2 & b_2 & c_2 & 0 & \cdots & 0 \\ 0 & a_3 & b_3 & c_3 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & a_{n-1} & b_{n-1} & c_{n-1} \\ 0 & 0 & 0 & 0 & a_n & b_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ \vdots \\ r_{n-1} \\ r_n \end{bmatrix}.$$

Then

$$\boldsymbol{L} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ d_2 & 1 & 0 & 0 & \cdots & 0 \\ 0 & d_3 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & d_{n-1} & 1 & 0 \\ 0 & 0 & 0 & 0 & d_n & 1 \end{bmatrix}$$

$$\boldsymbol{U} = \begin{bmatrix} e_1 & f_1 & 0 & 0 & \cdots & 0 \\ 0 & e_2 & f_2 & 0 & \cdots & 0 \\ 0 & 0 & e_3 & f_3 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & e_{n-1} & f_{n-1} \\ 0 & 0 & 0 & 0 & 0 & e_n \end{bmatrix}$$

Then

$$c_i = f_i, \ \ i = 1, \cdots, n-1$$

$$b_i = e_i + f_{i-1} d_i, \ \ i = 2, \cdots, n \qquad b_1 = e_1$$

$$a_i = d_i e_{i-1}, \ \ i = 2, \cdots, n.$$

Then

$$f_i = c_i, \ \ i = 1, \cdots, n-1$$

$$d_i = \frac{a_i}{e_{i-1}}, \ \ i = 2, \cdots, n$$

$$e_i = b_i - f_{i-1} d_i = b_i - f_{i-1} \frac{a_i}{e_{i-1}}, \ \ i = 2, \cdots, n.$$

3.3 Round-Off Error - The Condition Number

The round-off error is unavoidable in computer calculation.

Question: Can we roughly detect how much round-off errors are in the computed solution?

Answer: Yes. Compute the condition number.

Let δb be the round-off error in b and δx be the corresponding error in x. Then

$$\boldsymbol{A}(x + \delta x) = b + \delta b$$

$$\Downarrow$$

$$\delta x = \boldsymbol{A}^{-1} \delta b.$$

$$\Downarrow$$

$$||\delta x|| \leq ||\boldsymbol{A}^{-1}|| ||\delta b||.$$

Since $||b|| \leq ||\boldsymbol{A}|| ||x||$,

$$||\delta x|| ||b|| \leq ||\boldsymbol{A}^{-1}|| ||\delta b|| ||\boldsymbol{A}|| ||x||.$$

$$\Downarrow$$

$$\frac{\frac{||\delta x||}{||x||}}{\frac{||\delta b||}{||b||}} \leq ||\boldsymbol{A}|| ||\boldsymbol{A}^{-1}|| =: cond(\boldsymbol{A})$$

where $cond(\boldsymbol{A})$ is called the condition number.

Similarly if δA is the round-off error in A, then

$$\frac{\frac{||\delta x||}{||x||}}{\frac{||\delta A||}{||A||}} \leq cond(A).$$

In LU decomposition, $LU \approx A$.

In Gaussian elimination, the triangular matrix and the corresponding b vector are not exact.

Hence, for accurate computation, it is highly desirable $cond(A) \ll 1$.

The calculation of A^{-1} to compute $cond(A)$ is infeasible. Instead of computing exact $cond(A)$, a common alternative is

$$cond(A) \approx \frac{|||A|||_2}{|det(A)|}.$$

Question: What can we do if $cond(A) \gg 1$?

Answer: Improve the solution using the iterative techniques in the next section.

3.4 Iterative Techniques

Jacobi Iteration

Notice that

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ -a_{21} & 0 & 0 \\ -a_{31} & -a_{32} & 0 \end{bmatrix} - \begin{bmatrix} 0 & -a_{12} & -a_{13} \\ 0 & 0 & -a_{23} \\ 0 & 0 & 0 \end{bmatrix} = D - L - U.$$

Then

$$[D - L - U]x = b$$

Jacobi Iteration: Starting from the initial guess x^0,

$$x^{m+1} = D^{-1}[b + (L+U)x^m]$$

or

$$x_i^{m+1} = \frac{1}{a_{ii}} \left[b_i - \sum_{j=1, j\neq i}^{n} a_{ij} x_j^m \right].$$

Question: Does Jacobi interation converges to the solution of $Ax = b$.

Answer: From the contraction mapping theorem, the iteration converges if $||D^{-1}(L+U)||_\infty < 1$. However,

$$D^{-1}(L+U) = \begin{bmatrix} \frac{1}{a_{11}} & 0 & 0 \\ 0 & \frac{1}{a_{22}} & 0 \\ 0 & 0 & \frac{1}{a_{33}} \end{bmatrix} \begin{bmatrix} 0 & -a_{12} & -a_{13} \\ -a_{21} & 0 & -a_{23} \\ -a_{31} & -a_{32} & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{a_{12}}{a_{11}} & -\frac{a_{13}}{a_{11}} \\ -\frac{a_{21}}{a_{22}} & 0 & -\frac{a_{23}}{a_{22}} \\ -\frac{a_{31}}{a_{33}} & -\frac{a_{32}}{a_{33}} & 0 \end{bmatrix}.$$

Then $||D^{-1}(L+U)||_\infty < 1$ reduces to

$$\max_{i=1,\cdots,n} \left(\frac{1}{|a_{ii}|} \sum_{j=1, j\neq i}^{n} |a_{ij}| \right) < 1$$

or

$$|a_{ii}| > \sum_{j=1, j\neq i}^{n} |a_{ij}|.$$

This is the so called diagonal dominance condition for convergence.

Gauss-Seidel Iteration

Guass-Seidel iteration is a variant of Jacobi iteration where the newest possible estimate is used to hopefully improve the convergence speed.

Gauss-Seidel iteration:

$$x_i^{m+1} = \frac{1}{a_{ii}} \left[b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{m+1} - \sum_{j=i+1}^{n} a_{ij} x_j^m \right].$$

Guass-Seidel iteration converges faster than the Jacobi iteration if it converges. However, it is more likely to fail to converge.

To attain the convergence or to enhance the convergence speed, the relaxation is often employed.

$$x_i^{(m+1)} = \omega x_i^{(m+1)} + (1-\omega) x_i^{(m)} = x_i^{(m)} + \omega(x_i^{(m+1)} - x_i^{(m)}).$$

- $0 < \omega < 1$: underrelaxation (better stability than Gauss-Seidel)

- $1 < \omega < 2$: overrelaxation (faster convergence speed than Gauss-Seidel)

  Repeated application of overrelaxtion is called successive overrelaxation (SOR) that is most popular.

- $\omega \geq 2$: diverges

3.5 Eigensystems

Let A be Hermitian. Let λ_j be an eigenvalue of A and v_j be the corresponding eigenvector. Then

$$v_j^T A v_j = \lambda_j v_j^T v_j$$

or

$$\lambda_j = \frac{v_j^T A v_j}{v_j^T v_j}.$$

This equation is called Rayleigh quotient.

Goal: Find the largest eigenvalue and the corresponding eigenvector.

Consider

$$y = c_1 v_1 + \cdots + c_n v_n.$$

$$\Downarrow$$

$$Ay = c_1 A v_1 + \cdots + c_n A v_n = c_1 \lambda_1 v_1 + \cdots + c_n \lambda_n v_n.$$

$$\Downarrow$$

$$A^m y = \sum_{j=1}^{n} c_j \lambda_j^m v_j$$

Suppose λ_1 is the largest eigenvalue. Then for $m \gg 1$,

$$A^m y \approx c_1 \lambda_1^m v_1.$$

Hence, for $m \gg 1$, let

$$v_1^{(m)} = A^m y$$

and

$$\lambda_1^{(m)} \approx \frac{(v_1^{(m)})^T A v_1^{(m)}}{(v_1^{(m)})^T v_1^{(m)}} = \frac{(v_1^{(m)})^T A v_1^{(m)}}{||v_1^{(m)}||^2}.$$

Part II

Vector Calculus

Chapter 4

Vector Differential Calculus

4.1 Derivatives of Vector Function

Convergence of vector sequence: $\{v_i\}_{i=1}^{\infty}$ is said to converge if there exists l such that

$$\lim_{i\to\infty} ||v_i - l|| = 0.$$

l is called the limit and we write

$$\lim_{i\to\infty} v_i = l.$$

Ex: Consider the vector sequence

$$\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ \frac{2}{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{4} \\ \frac{2}{4} \end{bmatrix}, \begin{bmatrix} \frac{1}{6} \\ \frac{2}{6} \end{bmatrix}, \cdots \right\}.$$

Then the limit is

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Convergence of vector function ($f : \mathbf{R} \to \mathbf{R}^n$): $v(t)$ is said to converge as $t \to t_0$ if there exists l such that

$$\lim_{t\to t_0} ||v(t) - l|| = 0.$$

l is called the limit and we write

$$\lim_{t\to t_0} v(t) = l.$$

Continuity of vector function: $v(t)$ is said to be continuous at t_0 if

$$\lim_{t \to t_0} v(t) = v(t_0).$$

Derivative of vector function: $v(t)$ is differentiable at t_0 if

$$\lim_{\alpha \to 0} \frac{v(t_0 + \alpha) - v(t_0)}{\alpha} \quad \text{exists}$$

$$\Updownarrow$$

$\exists \eta(t_0) \in \mathbf{R}$ such that

$$\lim_{\alpha \to 0} \frac{v(t_0 + \alpha) - v(t_0) - \alpha\eta(t_0)}{\alpha} = 0.$$

$\eta(t_0)$ is called the derivative of v, denoted $\frac{dv(t_0)}{dt}$ or $v'(t_0)$.

In Cartesian coordinate,

$$v'(t_0) = \begin{bmatrix} v'_1(t_0) \\ v'_2(t_0) \\ \vdots \\ v'_n(t_0) \end{bmatrix}.$$

Fact: $(cv(t))' = cv'(t)$.

Proof: If $c = 0$, trivial. Suppose $c \neq 0$. Then

$$0 = \lim_{\alpha \to 0} \frac{cv(t + \alpha) - cv(t) - \alpha(cv(t))'}{\alpha} = c \lim_{\alpha \to 0} \frac{v(t + \alpha) - v(t) - \alpha\frac{(cv(t))'}{c}}{\alpha}.$$

Product Rules:

- $(u \cdot v)' = u' \cdot v + u \cdot v'$
- $(u \times v)' = u' \times v + u \times v'$
- $[u \cdot (v \times w)]' = u' \cdot (v \times w) + u \cdot (v' \times w) + u \cdot (v \times w')$

4.2 Derivatives of Scalar Fields

Gradient

Scalar field: $f : \mathbf{R}^n \to \mathbf{R}$.

Definition: $f : \mathbf{R}^n \to \mathbf{R}$ is differentiable at $\bar{x} \in \mathbf{R}^n$ if $\exists \eta(\bar{x}) \in \mathbf{R}^n$ such that

$$\lim_{h \to 0} \frac{f(\bar{x}+h) - f(\bar{x}) - \eta^T(\bar{x})h}{||h||} = 0$$

where h is an n-D vector. $\eta(\bar{x})$ is called the gradient of f, denoted as $\nabla f(\bar{x})$.

Fact: If $f : \mathbf{R}^n \to \mathbf{R}$ is differentiable at $\bar{x}$, then $\eta(\bar{x})$ is unique.

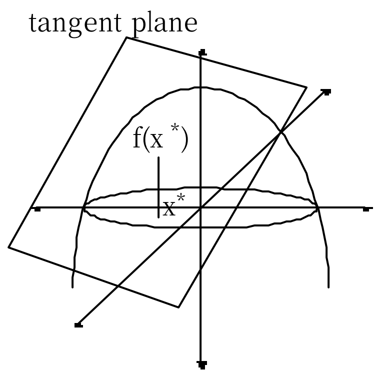
Proof: Suppose $\nu(\bar{x})$ is another gradient of f at $\bar{x}$. Let $h = \alpha e_i$. Then

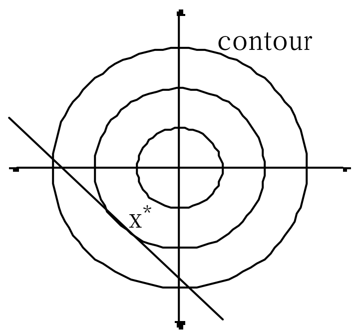
$$\lim_{\alpha \to 0} \frac{f(\bar{x}+\alpha e_i) - f(\bar{x})}{\alpha} = \eta_i(\bar{x}) = \nu_i(\bar{x}).$$

Hence, the fact follows.

Geometric Interpretation of Gradient:

Consider the tangent plane of f at $\bar{x}$. Then the direction of the gradient is the steepest ascent direction.





Equation for tangent plane:

$$f(x) = f(\bar{x}) + \nabla f(\bar{x})^T (x - \bar{x}) \quad \Rightarrow \quad \Delta f = \nabla f^T \Delta x$$

Note that $\frac{\triangle f}{|\triangle x|}$ is the slope in the direction of $\triangle x$. To find the steepest ascent direction, consider

$$\max_{|\triangle x|=1} \frac{\triangle f}{|\triangle x|} = \max_{|\triangle x|=1} \triangle f = \max_{|\triangle x|=1} \nabla f^T \triangle x$$

$$= \max_{|\triangle x|=1} |\nabla f||\triangle x| \cos(\angle(\nabla f, \triangle x)) = \max_{|\triangle x|=1} |\nabla f| \cos(\angle(\nabla f, \triangle x)).$$

Notice that the maximum is achieved when $\angle(\nabla f, \triangle x) = 0$.

Moreover, consider small $\triangle x$ in the tangent line of the contour. Then

$$0 = \triangle f = \nabla f^T \triangle x.$$

Hence, ∇f and $\triangle x$ are orthogonal.

Partial Derivatives

Definition: $f : \mathbf{R}^n \to \mathbf{R}$ is said to have a partial derivative at x with respect to x_i if

$$\lim_{\alpha \to 0} \frac{f(x + \alpha e_i) - f(x)}{\alpha} \quad \text{exists.}$$

$$\Updownarrow$$

$\exists\, \nu_i(x) \in \mathbf{R}$ such that

$$\lim_{\alpha \to 0} \frac{f(x + \alpha e_i) - f(x) - \alpha \nu_i(x)}{\alpha} = 0.$$

$\nu_i(x)$ is called the partial derivative of f w.r.t. x_i, denoted $\frac{\partial f(x)}{\partial x_i}$. Let

$$f_i(x_i) = f(x_1, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_n).$$

Then

$$\frac{\partial f}{\partial x_i}(x) = \frac{df_i}{dx_i}(x_i).$$

Fact: If $f : \mathbf{R}^n \to \mathbf{R}$ is differentiable at x, then f has partial derivatives at x.

Proof: Since f is differentiable at x,there is a vector $\eta(x)$ so that

$$\lim_{h \to 0} \frac{f(x + h) - f(x) - \eta^T(x) h}{||h||} = 0.$$

This is true no matter how the vector h approaches to 0. Therefore, let $h(\alpha) = \alpha e_i$ and then $h(\alpha) \to 0$ as $\alpha \to 0$. Then

$$\lim_{\alpha \to 0} \frac{f(x + \alpha e_i) - f(x) - \alpha \eta^T(x) e_i}{\alpha} = 0.$$

This implies, by uniqueness, that $\nu_i(x) = \eta^T(x) e_i = \eta_i(x)$. Hence $\eta_i(x) = \frac{\partial f(x)}{\partial x_i}$.

From the proof of this theorem,

$$\nabla f(x) = \begin{bmatrix} \frac{\partial f(x)}{\partial x_1} \\ \frac{\partial f(x)}{\partial x_2} \\ \vdots \\ \frac{\partial f(x)}{\partial x_n} \end{bmatrix}.$$

Directional Derivatives

Definition: $f : \mathbf{R}^n \to \mathbf{R}$ is said to be differentiable at x in the direction of d if

$$\lim_{\alpha \searrow 0} \frac{f(x + \alpha d) - f(x)}{\alpha ||d||} \quad \text{exists.}$$

When this limit exists, we will write it as $\boldsymbol{D}_d f(x)$.

Notice that

$$\lim_{\alpha \searrow 0} \frac{f(x + \alpha d) - f(x) - \alpha \boldsymbol{D}_d f(x) ||d||}{\alpha ||d||} = 0$$

Note: When $d = e_i$, we have $\boldsymbol{D}_d f(x) = \frac{\partial f(x)}{\partial x_i}$.

If $d_1 \neq 0$, let

$$f_d(x_1) = f(x_1, x_2 + \frac{d_2}{d_1}(x_1 - x_1), \cdots, x_n + \frac{d_n}{d_1}(x_1 - x_1)).$$

Then

$$\boldsymbol{D}_d f(x) = \frac{df_d}{dx_1}(x_1).$$

Fact: Let $f : \mathbf{R}^n \to \mathbf{R}$ be differentiable at x. Let $d \neq 0$ be any direction. Then

$$\boldsymbol{D}_d f(x) = \frac{\nabla f^T(x) d}{||d||}.$$

Proof: Since f is differentiable at x, we know that

$$\lim_{h \to 0} \frac{f(x+h) - f(x) - \nabla f^T(x)h}{||h||} = 0$$

no matter how h approach 0. Now let $h(\alpha) = \alpha d$; then $h(\alpha) \to 0$ as $\alpha \to 0$. Therefore,

$$\lim_{\alpha \searrow 0} \frac{f(x+\alpha d) - f(x) - \alpha \nabla f^T(x) d}{\alpha ||d||} = 0.$$

However,

$$\lim_{\alpha \searrow 0} \frac{f(x+\alpha d) - f(x) - \alpha D_d f(x) ||d||}{\alpha ||d||} = 0.$$

Therefore, by uniqueness, we must have

$$\nabla f^T(x) d = D_d f(x) ||d||.$$

Corollary: the norm of the gradient is the slope of the steepest ascent direction.

Proof:

$$D_{\nabla f(x)} f(x) = \frac{\nabla f^T(x) \nabla f(x)}{||\nabla f(x)||} = \frac{||\nabla f(x)||^2}{||\nabla f(x)||} = ||\nabla f(x)||.$$

4.3 Derivatives of Vector Fields

Jacobian

Vector field: $f : \mathbf{R}^n \to \mathbf{R}^n$.

Definition: $f : \mathbf{R}^n \to \mathbf{R}^n$ is differentiable at $x \in \mathbf{R}^n$ if $\exists \eta(x) \in \mathbf{R}^{n \times n}$ such that

$$\lim_{h \to 0} \frac{f(x+h) - f(x) - \eta(x) h}{||h||} = 0$$

where h is an n-D vector. $\eta(x)$ is called the Jacobian of f, denoted as $\nabla f(x)$.

Fact: If $f : \mathbf{R}^n \to \mathbf{R}^n$ is differentiable at x, then $\eta(x)$ is unique.

Fact: If $f : \mathbf{R}^n \to \mathbf{R}^n$ is differentiable at x, then f_i has partial derivatives at x and

$$\nabla f(x) = \begin{bmatrix} \frac{\partial f_1(x)}{\partial x_1} & \frac{\partial f_1(x)}{\partial x_2} & \cdots & \frac{\partial f_1(x)}{\partial x_n} \\ \frac{\partial f_2(x)}{\partial x_1} & \frac{\partial f_2(x)}{\partial x_2} & \cdots & \frac{\partial f_2(x)}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f_n(x)}{\partial x_1} & \frac{\partial f_n(x)}{\partial x_2} & \cdots & \frac{\partial f_n(x)}{\partial x_n} \end{bmatrix} = \begin{bmatrix} \nabla f_1^T(x) \\ \nabla f_2^T(x) \\ \vdots \\ \nabla f_n^T(x) \end{bmatrix}.$$

Hessian: The gradient of a scalar field is a vector field and its Jacobian is called Hessian:

$$\boldsymbol{H}(x) = \frac{\partial^2 f}{\partial x^2} = \nabla[\nabla f](x) = \begin{bmatrix} \frac{\partial^2 f(x)}{\partial x_1^2} & \frac{\partial^2 f(x)}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f(x)}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f(x)}{\partial x_2 \partial x_1} & \frac{\partial^2 f(x)}{\partial x_2^2} & \cdots & \frac{\partial^2 f(x)}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial^2 f(x)}{\partial x_n \partial x_1} & \frac{\partial^2 f(x)}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f(x)}{\partial x_n^2} \end{bmatrix}.$$

Indeed, the Hessian is a second derivative of a scalar field.

Note: Hessian must be distinguished from the Laplacian:

$$\nabla^2 f = \nabla \cdot \nabla f = \frac{\partial^2 f}{\partial x_1^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2}.$$

Divergence

Divergence:

$$\nabla \cdot f = \frac{\partial f_1}{\partial x_1} + \cdots + \frac{\partial f_n}{\partial x_n}.$$

Notice that the Laplacian is the divergence of the gradient of f.

Curl

Curl:

$$\nabla \times f = det \begin{bmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{bmatrix}$$

$$= \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) i + \left(\frac{\partial f_1}{\partial y} - \frac{\partial f_3}{\partial x}\right) j + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) k = \begin{bmatrix} \frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \\ \frac{\partial f_1}{\partial y} - \frac{\partial f_3}{\partial x} \\ \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \end{bmatrix}.$$

Chapter 5

Vector Integral Calculus

5.1 Line Integral

Smooth Curve C: for $a \leq t \leq b$,

$$r(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$$

where r is a differentiable vector function.

Line Integral of a vector field $\boldsymbol{F}$ over a smooth curve C:

$$\int_C \boldsymbol{F}(r) \cdot dr = \int_a^b \boldsymbol{F}(r(t)) \cdot \frac{dr}{dt} dt$$

or

$$\int_C (F_1 dx + F_2 dy + F_3 dz) = \int_a^b (F_1 x' + F_2 y' + F_3 z') dt.$$

If C is a closed curve, we use $\oint_C$ instead of $\int_C$.

Special cases of line integral:

- When $\boldsymbol{F} = F_1 e_1, F_2 e_2, F_3 e_3$,

$$\int_C F_1 dx, \quad \int_C F_2 dy, \quad \int_C F_3 dz.$$

- When $\boldsymbol{F} = \frac{G}{x'} e_1$,

$$\int_C G(r(t)) dt = \int_a^b G(r(t)) dt.$$

Between two end points of a curve, there exist numerous curves. In general, the line integrals for different paths are different. However, in some cases, the line integral is independent of path. Indeed, we have the following theorems.

Theorem: The line integral is independent of path iff

$$\boldsymbol{F} = \nabla f$$

or

$$F_1 = \frac{\partial f}{\partial x}, \quad F_2 = \frac{\partial f}{\partial y}, \quad F_3 = \frac{\partial f}{\partial z}.$$

Proof: ($\Leftarrow$) Let C be any path from any point $\boldsymbol{A}$ to any point $\boldsymbol{B}$, given by

$$r(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}, \quad a \leq t \leq b.$$

Then by chain rule,

$$\int_C (F_1 dx + F_2 dy + F_3 dz) = \int_C \left(\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz \right)$$

$$= \int_C df = \int_a^b \frac{df}{dt} dt = f(x(t), y(t), z(t))|_{t=a}^{t=b} = f(\boldsymbol{B}) - f(\boldsymbol{A}).$$

($\Rightarrow$) Let $\boldsymbol{A} = (x_0, y_0, z_0)$ and $\boldsymbol{B} = (x, y, z)$. Define

$$f(x, y, z) = f_0 + \int_C (F_1 dx^* + F_2 dy^* + F_3 dz^*)$$

with any constant f_0 and any path C from $\boldsymbol{A}$ to $\boldsymbol{B}$. We may integrate along a path C_1 from $\boldsymbol{A}$ to $\boldsymbol{B}_1 = (x_1, y, z)$ and along the path C_2 from $\boldsymbol{B}_1$ to $\boldsymbol{B}$ parallel to x axis. Then

$$f(x, y, z) = f_0 + \int_{C_1} (F_1 dx^* + F_2 dy^* + F_3 dz^*) + \int_{C_2} (F_1 dx^* + F_2 dy^* + F_3 dz^*).$$

$$= f_0 + \int_{C_1} (F_1 dx^* + F_2 dy^* + F_3 dz^*) + \int_{x_1}^{x} F_1(x^*, y, z) dx^*.$$

Then

$$\frac{\partial f}{\partial x} = F_1(x, y, z).$$

Similarly,
$$\frac{\partial f}{\partial y} = F_2(x,y,z), \quad \frac{\partial f}{\partial z} = F_3(x,y,z).$$

The differential form

$$\boldsymbol{F} \cdot dr = F_1 dx + F_2 dy + F_3 dz$$

is said to be exact if
$$\boldsymbol{F} = \nabla f$$

such that
$$F_1 dx + F_2 dy + F_3 dz = df.$$

Hence, the line integral is independent of path iff the differential form is exact.

Theorem: The line integral is independent of path iff its value around every closed path is zero.

Proof: ($\Rightarrow$) Since the line integral is independent of path, the line integrals from $\boldsymbol{A}$ to $\boldsymbol{B}$ along C_1 and along C_2 give the same value. Notice that $C = C_1 \cup C_2$ is a closed curve. If we integrate from $\boldsymbol{A}$ to $\boldsymbol{B}$ along C_1 and from $\boldsymbol{B}$ to $\boldsymbol{A}$ along C_2, the sum of two integrals is zero.

($\Leftarrow$) Given any points $\boldsymbol{A}$ and $\boldsymbol{B}$, let C be any closed curve passing through $\boldsymbol{A}$ and $\boldsymbol{B}$. Let C_1 be a part of C from $\boldsymbol{A}$ and $\boldsymbol{B}$ and C_2 the rest. Since the line integral around C is zero, the line integrals from $\boldsymbol{A}$ to $\boldsymbol{B}$ along C_1 and along C_2 must have the same value.

Theorem: The line integral is independent of path iff

$$\nabla \times \boldsymbol{F} = 0$$

or
$$\frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z}, \quad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \quad \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}.$$

Proof: ($\Rightarrow$) Since $\boldsymbol{F} = \nabla f$,

$$\nabla \times \boldsymbol{F} = \nabla \times (\nabla f) = \begin{bmatrix} \frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y} \\ \frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z} \\ \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \end{bmatrix} = 0.$$

(⇐) The proof requires the Stokes' theorem and thus, will be given in Stokes' theorem section.

Notice that in 2-D case, $\nabla \times F = 0$ reduces to

$$\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}.$$

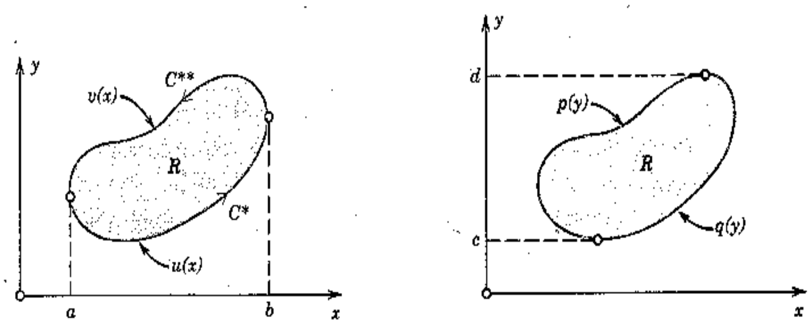
5.2 Green's Theorem in the Plane

Green's theorem in the plane (Transformation between double integrals and line integrals): Let $R \subset \mathbf{R}^2$ be a closed bounded region whose boundary ∂R consists of finitely many smooth curves. Let F_1, F_2 be continuously differentiable. Then

$$\int\int_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy = \oint_{\partial R} (F_1 dx + F_2 dy)$$

where line integral is done in the direction such that R is on the left as we advance.

Proof:



We first prove Green's theorem for a special region R that can be represented in both the forms

$$a \leq x \leq b, \quad u(x) \leq y \leq v(x)$$

and

$$c \le y \le d, \quad p(y) \le x \le q(y).$$

Notice that

$$\int\int_R \frac{\partial F_1}{\partial y} dx dy = \int_a^b \left[\int_{u(x)}^{v(x)} \frac{\partial F_1}{\partial y} dy \right] dx$$

$$= \int_a^b \left[F_1(x, v(x)) - F_1(x, u(x)) \right] dx = -\int_a^b F_1(x, u(x)) dx - \int_b^a F_1(x, v(x)) dx$$

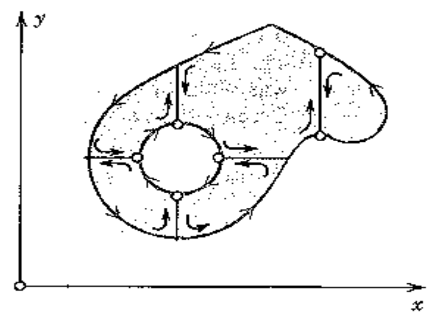
$$= -\int_{C'} F_1(x,y) dx - \int_{C''} F_1(x,y) dx = -\oint_{\partial R} F_1(x,y) dx.$$

Similarly,

$$\int\int_R \frac{\partial F_2}{\partial x} dx dy = \oint_{\partial R} F_2(x,y) dy.$$

Hence, the theorem for special region follows.

In general, split the R into several pieces of special regions.



5.3 Surface Integral

Surfaces

Representation of a surface S:

- Explicit form: $z = f(x, y)$.
- Implicit form: $g(x, y, z) = 0$.

  Note that explicit form is a special case of implicit form ($f(x, y) - z = 0$).
- Parametric form:

$$r(u, v) = \begin{bmatrix} x(u, v) \\ y(u, v) \\ z(u, v) \end{bmatrix} \quad (u, v) \in \boldsymbol{R}$$

  Note that explicit form is a special case of parametric form $\left(r = \begin{bmatrix} x \\ y \\ f(x, y) \end{bmatrix} \right)$.

Tangent Plane and Surface Normal

Suppose the surface S is given by the implicit form. Consider any curve $r(t)$ on S. Then

$$0 = \frac{dg}{dt} = \nabla g \cdot \dot{r}.$$

Since $\dot{r}$ is in the tangent plane, ∇g is normal to S. Hence, the unit normal vector is

$$n = \frac{\nabla g}{||\nabla g||}.$$

Suppose the surface S is given by the parametric form. Consider any curve $(u(t), v(t))$ in $\boldsymbol{R}$. Then

$$\frac{dr}{dt} = r_u u' + r_v v'$$

where

$$r_u = \frac{\partial r}{\partial u} \quad r_v = \frac{\partial r}{\partial v}.$$

Since the tangent plane is spanned by r_u and r_v, the unit normal vector is

$$n = \frac{r_u \times r_v}{||r_u \times r_v||} = \frac{\boldsymbol{N}}{||\boldsymbol{N}||}$$

where

$$\boldsymbol{N} = r_u \times r_v.$$

Surface Integral

Surface integral of a vector function $\boldsymbol{F}$ over S:

$$\int\int_S \boldsymbol{F}\cdot n d\boldsymbol{A} = \int\int_R \boldsymbol{F}[r(u,v)]\cdot \boldsymbol{N}(u,v)dudv.$$

Notice that since $||\boldsymbol{N}|| = ||r_u \times r_v||$ is the area of the parallelogram formed by r_u and r_v,

$$d\boldsymbol{A} = d[||(dur_u)\times(dvr_v)||] = ||r_u\times r_v||dudv = ||\boldsymbol{N}||dudv.$$

Moreover,

$$\int\int_S \boldsymbol{F}\cdot n d\boldsymbol{A} = \int\int_S (\boldsymbol{F}_1(i\cdot n) + \boldsymbol{F}_2(j\cdot n) + \boldsymbol{F}_3(k\cdot n))d\boldsymbol{A}$$
$$= \int\int_S (\boldsymbol{F}_1\cos\alpha + \boldsymbol{F}_2\cos\beta + \boldsymbol{F}_3\cos\gamma)d\boldsymbol{A}$$

where α, β, γ are the angles between n and positive axes.

If n is a normal vector, so is $-n$. Indeed, if we interchange u and v, $r_v \times r_u = -r_u \times r_v = -\boldsymbol{N}$. Hence, the integral depends on the choice of unit normal vector. Such an integral is called an integral over an oriented surface.

Suppose the surface can be defined by

$$x = f_x(y,z), \quad (y,z)\in \boldsymbol{R}_x.$$

Then

$$\int\int_S \boldsymbol{F}_1\cos\alpha d\boldsymbol{A} = \int\int_S \boldsymbol{F}_1 dydz$$
$$= \begin{cases} \int\int_{\boldsymbol{R}_x} \boldsymbol{F}_1(f_x(y,z),y,z)dydz & \text{if } \cos\alpha > 0 \\ -\int\int_{\boldsymbol{R}_x} \boldsymbol{F}_1(f_x(y,z),y,z)dydz & \text{if } \cos\alpha < 0 \end{cases}.$$

We have similar results for $\boldsymbol{F}_2$ and $\boldsymbol{F}_3$. Hence,

$$\int\int_S \boldsymbol{F}\cdot n d\boldsymbol{A} = \int\int_S (\boldsymbol{F}_1 dydz + \boldsymbol{F}_2 dzdx + \boldsymbol{F}_3 dxdy).$$

Integrals over a nonoriented surfaces:

$$\int\int_S G(r)dA = \int\int_R G[r(u,v)]||N(u,v)||dudv.$$

If S is given by $z = f(x,y)$, let $u = x, v = y, r = [u\ v\ f]^T$. Then

$$||N(u,v)|| = ||r_u \times r_v|| = ||[1\ 0\ f_u]^T \times [0\ 1\ f_v]^T||$$

$$= ||[-f_u\ -f_v\ 1]^T|| = \sqrt{1 + f_u^2 + f_v^2}.$$

Hence,

$$\int\int_S G(r)dA = \int\int_{R^*} G[x, y, f(x,y)]\sqrt{1 + f_u^2 + f_v^2}dxdy$$

where R^* is the projection of S onto the xy plane.

5.4 Gauss' Divergence Theorem

Gauss' Divergence Theorem (Transformation between volume integrals and surface integrals): Let T be a closed bounded region in $\mathbf{R}^3$ and ∂T be its surface. Then

$$\int\int\int_T \nabla \cdot F dV = \int\int_{\partial T} F \cdot n dA$$

or

$$\int\int\int_T \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dxdydz$$

$$= \int\int_{\partial T} (F_1 \cos\alpha + F_2 \cos\beta + F_3 \cos\gamma)\, dA$$

where n is the outer unit normal vector of ∂T.

Proof: We first prove

$$\int\int\int_T \frac{\partial F_3}{\partial z} dxdydz = \int\int_{\partial T} F_3 \cos\gamma dA$$

for special region T represented in the form

$$g(x,y) \le z \le h(x,y)$$

where (x, y) varies in the projection $\boldsymbol{R}$ of $\boldsymbol{T}$ on xy plane. Notice that

$$\int\int\int_T \frac{\partial F_3}{\partial z} dxdydz = \int\int_R \left[\int_{g(x,y)}^{f(x,y)} \frac{\partial F_3}{\partial z} dz\right] dxdy$$

$$= \int\int_R [F_3[x, y, h(x, y)] - F_3[x, y, g(x, y)]] \, dxdy$$

$$= \int\int_R F_3[x, y, h(x, y)] dxdy - \int\int_R F_3[x, y, g(x, y)] dxdy$$

$$= \int\int_{\partial T} F_3 dxdy = \int\int_{\partial T} F_3 \cos\gamma dA.$$

Similarly,

$$\int\int\int_T \frac{\partial F_1}{\partial x} dxdydz = \int\int_{\partial T} F_1 \cos\alpha dA,$$

$$\int\int\int_T \frac{\partial F_2}{\partial y} dxdydz = \int\int_{\partial T} F_2 \cos\beta dA.$$

Hence, the theorem for special region follows.

In general, split the T into several pieces of special regions.

5.5 Stokes' Theorem

Stokes' Theorem (Transformation between surface integrals and line integrals): Let S be a piecewise smooth oriented surface and ∂S is piecewise smooth closed curve. Then

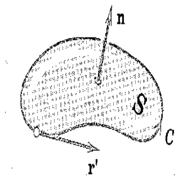
$$\int\int_S (\nabla \times \boldsymbol{F}) \cdot \boldsymbol{n} dA = \oint_{\partial S} \boldsymbol{F} \cdot \boldsymbol{r}'(s) ds$$

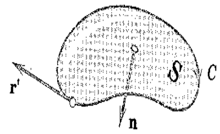
or

$$\int\int_R \left[\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) N_1 + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) N_2 + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) N_3\right] dudv$$

$$= \oint_{\partial R} (F_1 dx + F_2 dy + F_3 dz)$$

where n is a unit normal vector of S and r' is the unit tangent vector of ∂S such that





Moreover, $\boldsymbol{R}$ is the region in (u, v) corresponding to S represented by $h(u, v)$ and $\boldsymbol{N} = h_u \times h_v$.

Proof: We first prove

$$\int\int_R \left(\frac{\partial \boldsymbol{F}_1}{\partial z}\boldsymbol{N}_2 - \frac{\partial \boldsymbol{F}_1}{\partial y}\boldsymbol{N}_3\right) du dv = \oint_{\partial R} \boldsymbol{F}_1 dx$$

for special surface S represented in the form

$$z = f(x, y).$$

Setting $u = x, v = y$,

$$h(u, v) = h(x, y) = \begin{bmatrix} x \\ y \\ f(x, y) \end{bmatrix}$$

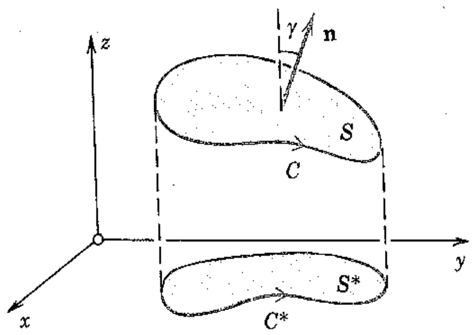
and thus

$$\boldsymbol{N} = h_u \times h_v = h_x \times h_y = \begin{bmatrix} -f_x \\ -f_y \\ 1 \end{bmatrix}.$$

Notice that $\boldsymbol{R}$ is the projection S^* of S onto the xy plane with $\partial \boldsymbol{R} = \boldsymbol{C}^*$.

Hence,

$$\int\int_R \left(\frac{\partial \boldsymbol{F}_1}{\partial z}\boldsymbol{N}_2 - \frac{\partial \boldsymbol{F}_1}{\partial y}\boldsymbol{N}_3\right) du dv = \int\int_{SR} \left(\frac{\partial \boldsymbol{F}_1}{\partial z}(-f_y) - \frac{\partial \boldsymbol{F}_1}{\partial y}\right) du dv$$



$$= \int\int_R -\frac{\partial F_1}{\partial y} dx dy = \oint_{\partial R} F_1 dx.$$

The second equality follows from the chain rule:

$$\frac{\partial F_1[x, y, f(x, y)]}{\partial y} = \frac{\partial F_1[x, y, z]}{\partial y} + \frac{\partial F_1[x, y, z]}{\partial x}\frac{\partial f}{\partial y}.$$

Similarly,

$$\int\int_R \left(-\frac{\partial F_2}{\partial z} N_1 + \frac{\partial F_2}{\partial x} N_3\right) du dv = \oint_{\partial R} F_2 dy,$$

$$\int\int_R \left(\frac{\partial F_3}{\partial y} N_1 - \frac{\partial F_3}{\partial x} N_2\right) du dv = \oint_{\partial R} F_3 dz.$$

Hence, the theorem for special region follows.

In general, split the T into several pieces of special regions.

Remark: Stokes' theorem is a generalization of Green's theorem.