비정상 상태의 축대칭형 자유 경계면 문제에 대한 수치 해석법 개발

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Development of a Numerical Scheme for Axisymmetric Unsteady Free-Boundary Problem

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Introduction

There have been remarkable progresses in development of numerical schemes for fluid flow anaylsis during the last several decades. However, one of the relatively underdeveloped areas is the unsteady free boundary problem in 2-dimensions. The numerical schemes for the free boundary problem analysis are mostly based on the finite element method (FEM). Although FEM is applicable to wide range of problems, it is known to require heavy computation time. Thus we propose an alternative scheme based on the finite difference method (FDM) on the numerically generated orthogonal coordinate system. In view of time differencing, the proposed scheme is explicit.

As a sample problem, the problem of pulsatile flows in an elastic blood vessel is considered. Time-periodic change of inlet velocity and pressure produces unsteadiness of the blood vessel wall as well as the blood flow. Thus a moving coordinate system is required, in other words the coordinate system depends on time. In the present work, an orthogonal coordinate system is generated for each time by the scheme of Oh and Kang[1]. Using the developed scheme, the pressure wave propagation phenomena in artery are studied.

Problem Description

1. Governing Equations in Orthogonal Coordinate System

Governing equations for blood flow are assumed to be given by the Navier-Stokes equation and the continuity equation. In this work, the stream function-vorticity formulation is adopted. The dimensionless governing equations for vorticity and stream function in an orthogonal coordinate system such as one in Figure 1 are given by [2]

$$\mathrm{St}\left(\frac{\partial \omega}{\partial t}\right)_{\mathbf{x}} + \frac{1}{h_\eta h_\xi}\left[\frac{\partial \psi}{\partial \xi}\frac{\partial}{\partial \eta}\left(\frac{\omega}{\sigma}\right) - \frac{\partial \psi}{\partial \eta}\frac{\partial}{\partial \xi}\left(\frac{\omega}{\sigma}\right)\right] = \frac{1}{\mathrm{Re}}L^2(\omega\sigma), \qquad (1)$$

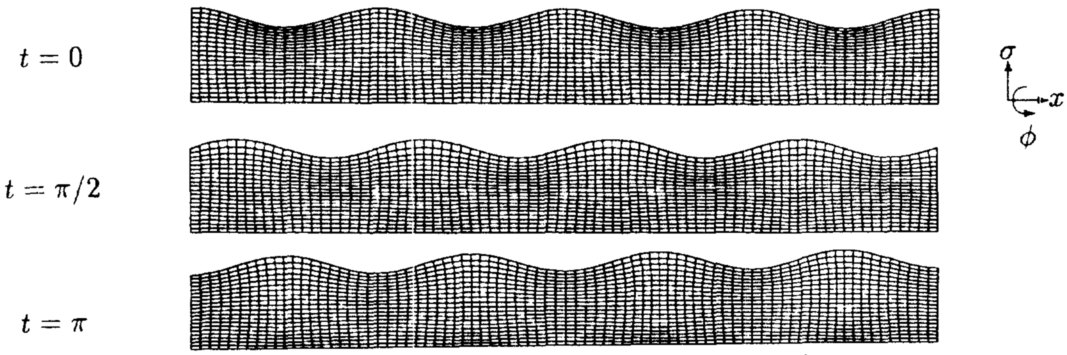

Figure 1: Moving coordinate system for an elastic tube with wave propagation

$$L^2\psi = \omega, \tag{2}$$

where

$$L^2 = \frac{1}{h_\eta h_\xi}\left[\frac{\partial}{\partial\xi}\left(\frac{f}{\sigma}\frac{\partial}{\partial\xi}\right) + \frac{\partial}{\partial\eta}\left(\frac{1}{f\sigma}\frac{\partial}{\partial\eta}\right)\right]$$

and the stream function is defined by

$$u_\xi = -\frac{1}{h_\eta\sigma}\frac{\partial\psi}{\partial\eta}, \quad u_\eta = \frac{1}{h_\xi\sigma}\frac{\partial\psi}{\partial\xi}\,. \tag{3}$$

In (1), the Strouhal number, $\mathrm{St} = \Omega l_c/u_c$, is the ratio of unsteady acceleration to the steady acceleration and is the measure of the unsteadiness of the flow. The Reynolds number, $\mathrm{Re} = u_c l_c/\nu$, is the ratio of the inertial force to the viscous force.

The shape of the physical domain is time-dependent as a result of deformation of the boundary, and the point in the physical domain corresponding to the fixed point (ξ, η) in the computational domain is also time-dependent. Transformation from the partial time derivatives at fixed points in the physical domain to the partial time derivatives in the (ξ, η) system must be done properly. Generally the relationship between the two time derivatives for an arbitrary dependent variable ω is given by

$$\left(\frac{\partial\omega}{\partial t}\right)_{\mathbf{x}} = \left(\frac{\partial\omega}{\partial t}\right)_{\boldsymbol{\xi}} - \left(\frac{\partial\mathbf{x}}{\partial t}\right)_{\boldsymbol{\xi}} \cdot \nabla\omega. \tag{4}$$

2. Boundary Conditions

In this problem, we assume that velocity and pressure are specified at inlet. As the inlet velocity condition, we use the transient uni-directional flow of time-dependent motion produced in a circular tube by a periodic, time-dependent pressure gradient

$$-\frac{\partial p}{\partial x} = G = G_0(1 + \epsilon \sin(\Omega t)) \ .$$

The flow field has time-independent part and time-dependent part and contains the Bessel functions of zero order. Nondimensionalized solution for $u_x = u_x^{(0)} + \epsilon\, u_x^{(1)}$ is given as

$$u_x^{(0)} = \frac{1}{4}\left(1 - r^2\right), \quad u_x^{(1)} = Re\left[\frac{e^{it}}{\mathrm{R}_\Omega}\left\{1 - \frac{J_0\left[\left(\frac{\mathrm{R}_\Omega}{i}\right)^{1/2} r\right]}{J_0\left[\left(\frac{\mathrm{R}_\Omega}{i}\right)^{1/2}\right]}\right\}\right] \ . \tag{5}$$

The osciallatory Reynolds number, or the pulsatile Reynolds number, $\mathrm{R}_\Omega = \Omega R_0^2/\nu$, represents the product of the Strouhal number and the Reynolds number. For simplicity, we use the same velocity distribution for the outlet condition as the inlet boundary condition. At tube wall, the condition for the stream function is derived from the kinematic condition.

3. Pressure Distribution and Tube Radius

From the solutions of vorticity and stream funcion, we can obtain the pressure distribution along the tube wall by using the relation

$$\begin{aligned} p &= -\frac{1}{2}(u_\xi^2 + u_\eta^2) \\ &+ \int_0^\xi \left[u_\eta \omega h_\xi - \frac{1}{\mathrm{Re}}\left(\frac{h_\xi}{h_\eta \sigma}\right)\frac{\partial(\sigma\omega)}{\partial \eta} - \mathrm{St}\, \mathbf{e}_\xi \cdot \left(\frac{\partial \mathbf{u}}{\partial t}\right)_{\mathbf{x}} h_\xi\right] d\xi + C(t), \end{aligned} \tag{6}$$

which is derived from the Navier-Stokes equation.

The relationship between the pressure distribution and the artery radius has been determined in experimental way[3]. The shape of tube wall is obtained by the relation

$$p(x,t) - P(R) = 0 \ . \tag{7}$$

Here, $P(R)$, which is determined experimentally, is the known function of the variation of R with pressure.

Numerical Strategy

The global numerical scheme is shown schematically in Figure 2. Starting from the grid system and the flow field solution at the n-th time step, the vorticity and the stream function for the $(n+1)$-st time step are obtained by solving (1) and (2) on the n-th time step grid system. The pressure distribution along the tube wall for the $(n+1)$-st time step is obtained by using (6) to predict the wall shape by (7). Then, the $(n+1)$-st time step grid system is generated. Since the coordinate system is changed, the variable correction according to (4) is required to obtain the values for the fixed points in the computational domain.

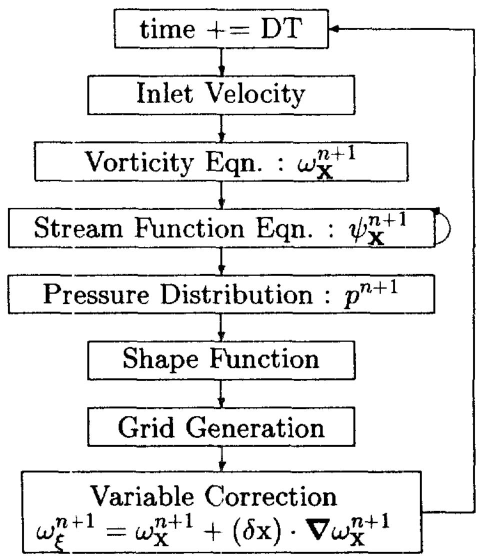

Figure 2: Flow diagram of numerical scheme

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