4장 상미분 방정식 — 경계치 문제

Boundary Value Problem

$$\mathcal{L}[\phi(x)] = g(x)$$

B.C. $\alpha_1 \phi(0) + \beta_1 \phi'(0) = a$

$\alpha_2 \phi(L) + \beta_2 \phi'(L) = b$

e.g. $\mathcal{L}[\phi(x)] = \left(\frac{d^2}{dx^2} + p\frac{d}{dx} + q\right)\phi$ linear prob.

$\mathcal{L}[\phi(x)] = \left(\frac{d^2}{dx^2} + p\frac{d}{dx} + q\right)\phi + D\phi^2$ non linear prob.

* 1. Uniqueness and existence is not known

2. Solution has "elliptic" character

* Type of methods

1. Initial-value method : 'Shooting method"
2. Finite difference method
3. Weighted residual method
   - Collocation
   - Galerkin
   - Least Square
   - Finite Element Method.

1. Shooting method.

(1) Linear Problem

Consider a linear 2-nd order BVP

$$\left(\gamma_1(x)\frac{d^2}{dx^2} + \gamma_2(x)\frac{d}{dx} + \gamma_3\right)\phi(x) = \mathcal{L}[\phi(x)] = g(x)$$

B.C. $\alpha_1\phi(0) + \beta_1\phi'(0) = a$

$\alpha_2\phi(1) + \beta_2\phi'(1) = b$

Decompose into 3 problems of IVP

(I) $\mathcal{L}(\phi_1(x)) = g(x)$ $\quad \phi_1(0) = \phi_1'(0) = 0$ $\quad$ f- solution

(II) $\mathcal{L}(\phi_2(x)) = 0$ $\quad \phi_2(0) = 1,\ \phi_2'(0) = 0$

(III) $\mathcal{L}(\phi_3(x)) = 0$ $\quad \phi_3(0) = 0,\ \phi_3'(0) = 1$

$>$ linearly independent

$$\phi(x) = \underbrace{a_1}_{=1}\phi_1(x) + a_2\phi_2(x) + a_3\phi(x)$$

to satisfy DE

Apply B.C.

$x=0$
$$a = \alpha_1[\cancel{\phi_1(0)} + a_2\underbrace{\phi_2(0)}_{=1} + \cancel{a_3\phi_3(0)}] + \beta_1[\cancel{\phi_1'(0)} + \cancel{a_2\phi_2'(0)} + a_3\underbrace{\phi_3'(0)}_{=1}]$$
$$= \alpha_1 a_2 + \beta_1 a_3$$

$x=1$
$$b = \alpha_2[\phi_1(1) + a_2\phi_2(1) + a_3\phi_3(1)] + \beta_2[\phi_1'(1) + a_2\phi_2'(1) + a_3\phi_3'(1)]$$

a: $\alpha_1 a_2 + \beta_1 a_3 = a$

$$[\alpha_2\phi_2(1) + \beta_2\phi_2'(1)]a_2 + [\alpha_2\phi_3(1) + \beta_2\phi_3'(1)]a_3 = b - \alpha_2\phi_1(1) - \beta_2\phi_1'(1)$$

Schemes for linear BVP

① Solve 3 IVP.

$$r_1(x)\frac{d^2\phi_i}{dx^2} + r_2(x)\frac{d'\phi_i}{dx^2} + r_3(x)\phi_i = g_i(x)$$

$$g_1(x) = g(x),\quad g_2(x) = 0,\quad g_3(x) = 0.$$

$$\phi_i(0) = u_i,\quad \phi_i'(0) = v_i$$

$$\frac{d\phi_i}{dx} = f_i \qquad \text{I.C. } \phi_i(0) = u_i$$

$$r_1(x)\frac{df_i}{dx} + r_2(x) f_i + r_3(x)\phi_i = g_i(x)$$

$$r_1(x)\frac{df_i}{dx} = -r_2(x) f_i - r_3(x)\phi_i + g_i(x)$$

$$\text{I.C. } f_i(0) = \phi_i'(0) = v_i$$

② Construct solution using B.C.

(2) Nonlinear Problem

$$\phi'' = \mathcal{L}[\phi] = f(\phi, \phi', x)$$

ex) $y'' = 2y^3 - 6y - 2x^3$

$y(1) = 2, \quad y(2) = \frac{5}{2}$

B.C. $\alpha_1 \phi(0) + \beta_1 \phi'(0) = a$

$\alpha_2 \phi(1) + \beta_2 \phi'(1) = b$

Let $\phi_1 = \phi$

$\phi_2 = \frac{d\phi}{dx}$

Then $\frac{d\phi_1}{dx} = \phi_2$

$\frac{d\phi_2}{dx} = f(\phi_1, \phi_2, x)$

unknown

Solve as a IVP $\phi_1(0) = v_1, \quad \phi_2(0) = v_2$

Look at B.C at $x=0$.

$\alpha_1 \phi_1(0) + \beta_1 \phi_2(0) = a$

$\alpha_1 v_1 + \beta_1 v_2 = a$

$$v_1 = \frac{a - \beta_2 v_2}{\alpha_1}$$

Iterative scheme

1. pick v_2 & set $v_1 = \frac{a - \beta_1 v_2}{\alpha_1}$
2. Integrate IVP to $x=1$.
3. Check B.C at $x=1$

$R(v_2) \equiv \alpha_2 \phi_1(1) + \beta_2 \phi_2(1) - b$

If $R(v_2) = 0$ Solution

If $R(v_2) \neq 0$ Go back to 1. & pick another v_2 and iterate

(3) Methods of Iteration for $R(v_2)=0$

— Successive Approximation

— Newton's method

1) Successive Approximation

$$v_2^{(k+1)} = v_2^{(k)} + R(v_2^{(k)}) \quad \text{Fixed point iteration}$$

Expect linear convergence

2) Newton's method

$$v_2^{(k+1)} = v_2^{(k)} - \frac{R(v_2^{(k)})}{\frac{dR}{dv_2}(v_2^{(k)})}$$

Evaluate $\frac{dR}{dv_2}(v_2^{(k)})$

Define the effect of a small change in v_2 on R

$$R(v_2+\delta v_2) = R(v_2) + \left(\frac{dR}{dv_2}\right)\delta v_2 + \cdots$$

$$\left.\frac{dR}{dv_2}\right|_{v_2^{(k)}} = \alpha_2\left[\frac{\partial \phi_1(1)}{\partial v_2}\right]_{v_2^{(k)}} + \beta_2\left[\frac{\partial \phi_2(1)}{\partial v_2}\right]_{v_2^{(k)}}$$

How do we evaluate

$$\left[\frac{\partial \phi_1(x)}{\partial v_2}\right]_{v_2^{(k)}} \equiv \dot{\phi}_1(x), \quad \left[\frac{\partial \phi_2(x)}{\partial v_2}\right]_{v_2^{(k)}} \equiv \dot{\phi}_2(x)$$

$$\phi_1(x; v_2+\delta v_2) = \phi_1(x; v_2) + \underbrace{\left[\frac{\partial \phi_1}{\partial v_2}\right]_{v_2}}_{\dot{\phi}_1(x)}\delta v_2$$

$$\phi_2(x; v_2+\delta v_2) = \phi_2(x; v_2) + \underbrace{\left[\frac{\partial \phi_2}{\partial v_2}\right]_{v_2}}_{\dot{\phi}_2(x)}\delta v_2$$

Differentiate ~~Original~~ DE wrt ~~v_2~~

$\frac{d\phi_1}{dx} = \phi_2 \rightarrow \frac{d\dot\phi_1}{dx} = \dot\phi_2$

$\frac{d\phi_2}{dx} = f$

$\frac{d\dot\phi_2}{dx} = \frac{\partial f}{\partial \phi_1}\frac{\partial\phi_1}{\partial v_2} + \frac{\partial f}{\partial\phi_2}\frac{\partial\phi_2}{\partial v_2}$

$\rightarrow \frac{d\dot\phi_2}{dx} = \frac{\partial f}{\partial\phi_1}\dot\phi_1 + \frac{\partial f}{\partial\phi_2}\dot\phi_2$

Substitute into original DE

$$\frac{d\phi_1}{dx} + \frac{d\dot\phi_1}{dx}\delta v_2 = \phi_2 + \dot\phi_2\,\delta v_2$$

$$\rightarrow \frac{d\dot\phi_1}{dx} = \dot\phi_2 \qquad (A1)$$

$$\frac{d\phi_2}{dx} + \frac{d\dot\phi_2}{dx}\delta v = f(\phi_1,\phi_2,x) + \frac{\partial f}{\partial\phi_1}(\phi_1,\phi_2,x)\,\dot\phi_1\,\delta v_2 + \frac{\partial f}{\partial\phi_2}(\phi_1,\phi_2,x)\,\dot\phi_2\,\delta v_2$$

$$\rightarrow \frac{d\dot\phi_2}{dx} = \frac{\partial f}{\partial\phi_1}(\phi_1,\phi_2,x)\,\dot\phi_1 + \frac{\partial f}{\partial\phi_2}(\phi_1,\phi_2,x)\,\dot\phi_2 \quad (A_2)$$

Solve (A1) & (A2) with IC's.

$\dot\phi_1(0) = ?\ ,\quad \dot\phi_2(0) = ?$

Diff wrt v_2

$\dot\phi_1(0) = -\frac{\beta_2}{\alpha_1}$

$\dot\phi_2(0) = 1$

$$\begin{cases}\phi_1(0) = v_1 \\ \phi_2(0) = v_2\end{cases} \xrightarrow{\text{Taylor series}} \begin{array}{l}\phi_1(0) + \dot\phi_1(0)\delta v_2 = v_1 \\ \phi_2(0) + \dot\phi_2(0)\,\delta v_2 = v_2 + \delta v_2\end{array}$$

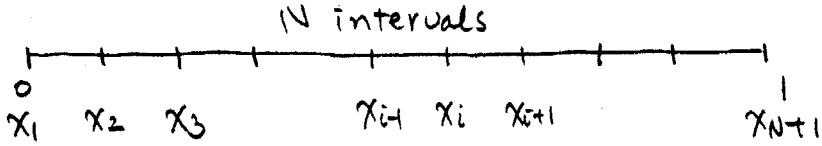
$$\dot\phi_1(0) = -\frac{\beta_2}{\alpha_1}, \quad \dot\phi_2(0) = 1$$

1. Solve for $\phi_1, \phi_2, \dot\phi_1, \dot\phi_2$ simultaneously
2. $\left.\frac{dR}{dv_2}\right|_{v_2^{(k)}} = \left[\alpha_2\dot\phi_1(1) + \beta_2\dot\phi_2(1)\right]_{v_2^{(k)}}$
3. $R(v_2) = \alpha_2\phi_1(1) + \beta_2\phi_2(1) - b$
4. $v_2^{(k+1)} = v_2^{(k)} - \frac{R(v_2^{(k)})}{\frac{dR}{dv_2}(v_2^{(k)})}$

2. Finite Difference Methods for BVP.

e.g. $\phi'' = f(\phi, \phi', x)$ with B.C

$0 \le x \le 1$



$h \equiv x_i - x_{i-1}$; equal spacing

Taylor series expansion

$$\phi_{i+1} \equiv \phi(x_{i+1}) = \phi_i + \phi_i' h + \phi_i'' \frac{h^2}{2} + \phi_i''' \frac{h^3}{3!} + \cdots$$

Forward series

$$\phi_{i-1} \equiv \phi(x_{i-1}) = \phi_i - \phi_i' h + \phi_i'' \frac{h^2}{2} + \phi_i''' \frac{h^3}{3!} + \cdots$$

backward series.

<u>Difference Formula</u>

Forward difference formula for $\frac{d\phi}{dx}$

$$\left.\frac{d\phi}{dx}\right|_{x_i} = \phi_i' = \frac{\phi_{i+1}-\phi_i}{h} - \frac{h}{2}\phi_i'' + O(h^2)$$

Backward difference formula

$$\left.\frac{d\phi}{dx}\right|_{x_i} = \phi_i' = \frac{\phi_i-\phi_{i-1}}{h} + \frac{h}{2}\phi_i'' + O(h^2)$$

} O(h) formula

Centered difference : Add two formulae and divide by two

$$\left.\frac{d\phi}{dx}\right|_{x_i} = \frac{\phi_{i+1}-\phi_{i-1}}{2h} - \frac{\phi_i''}{2\cdot 3!} h^2 + \cdots$$

$O(h^2)$ formula

Centered difference formula for $\frac{d^2\phi}{dx^2}$

Substract the two first derivative formulas

$$0 = \frac{\phi_i - \phi_{i-1}}{h} + \frac{h}{2}\phi_i'' - \frac{\phi_{i+1}-\phi_i}{h} + \frac{h}{2}\phi_i'' + O(h^3)$$

$$= h\phi_i'' + \frac{2\phi_i - \phi_{i+1} - \phi_{i-1}}{h} + O(h^3)$$

$$\phi_i'' = \frac{\phi_{i+1} - 2\phi_i + \phi_{i-1}}{h^2} + O(h^2)$$

Simple Problem

$$\phi'' = g(x) \qquad 0 \le x \le 1$$

$$\phi(0) = \phi(1) = 0$$

$\rightarrow \quad \underline{\underline{A}}\,\underline{x} = \underline{b}$

Properties of matrix $\underline{\underline{A}}$

At ith row, $\quad a_{ii} = \dfrac{-2}{h^2}$

$$a_{i,i-1} = a_{i,i+1} = \frac{1}{h^2}$$

$\Rightarrow$ Tridiagonal Matrix

Symmetric
Banded
Pos-def
Diagonally dominant

Take example from Finlayson

$$k = k_0 + k_0'(T - T_0)$$

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right) = 0, \quad T(0) = T_0, \quad T(1) = T_1$$

$$\theta = \frac{T - T_0}{T_1 - T_0} \qquad a = \frac{k_0'(T_1 - T_0)}{k_0}$$

Non dimensionalize

$$\frac{d}{dx}\left[(1 + a\theta)\frac{d\theta}{dx}\right] = 0 \qquad \theta(0) = 0, \quad \theta(1) = 1$$

$$(1 + a\theta)\frac{d^2\theta}{dx^2} + a\left(\frac{d\theta}{dx}\right)^2 = 0$$

Non-linear residual equation

$i = 1 \qquad \theta_i = 0$

$i = N+1 \qquad \theta_i = 1$

$i = 2 .. N \qquad \dfrac{\theta_{i+1} - 2\theta_i + \theta_{i-1}}{h^2} + \dfrac{a}{1 + a\theta_i}\left(\dfrac{\theta_{i+1} - \theta_{i-1}}{2h}\right)^2 = 0$

(N+1) eqs & (N+1) unknowns $\qquad$ centered difference

Take another set of BC's

$$\frac{d\theta}{dx}(0) = q,$$

$$\theta(1) = 1 \rightarrow \theta_{N+1} = 1$$

what's a good approximation for $\frac{d\theta}{dx}(0)$?

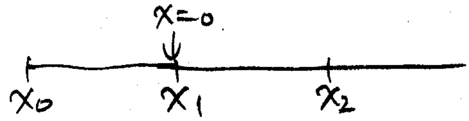
i) Forward difference

$$\frac{\theta_2 - \theta_1}{h} = q$$

Forward diff approximation is only $O(h)$!

ii) Centered difference . $O(h^2)$

create fictitious node.



$$\frac{d\theta}{dx}(0) = q \rightarrow \frac{\theta_2 - \theta_0}{2h} = q$$

(N+1) eqs & (N+2) unknowns.

do need another equation.

→ Use the differential equation at the boundary.

Define new eq set

$$R_0(\underline{\theta}) = \frac{\theta_2 - \theta_0}{2h} - q = 0$$

$$R_i(\underline{\theta}) = \frac{\theta_{i+1} - 2\theta_i + \theta_{i-1}}{h^2} + \frac{a}{(1+a\theta_i)}\left(\frac{\theta_{i+1} - \theta_{i-1}}{2h}\right)^2 = 0$$

$$i = 1 \cdots N$$

$$R_{N+1}(\underline{\theta}) = \theta_{N+1} - 1 = 0$$

(N+2) eqs & (N+2) unknowns.