Reading 4.2-4-5, 4-10

## 3. Weighted Residual Method

$$R(\theta) = \frac{d}{dx}\left[(1+a\theta)\frac{d\theta}{dx}\right] = 0$$

Weighted Residual Integral

$$\int_0^1 W_k(x)\,R(\theta)\,dx = 0 \qquad k = 1 \cdots N$$

2 Choices

1. Choice of $W_k(x)$
2. Choice of Approximate Functions

$$\theta(x) = \sum_{i=1}^{N} \theta_i \Phi^i(x)$$

$\theta_i$: coefficent unknown

$\Phi^i(x)$: basis function. known.

* Basis Functions

1. Global Functions

   defined on the entire domain

   may or may not be orthogonal

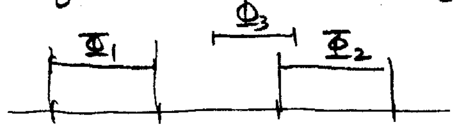
   (don't have to be orthogonal in numerical method)

   Orthogonal Collocation

   Spectral Methods (Useful in 1-D small prob) hard in 2-D

2. Local Functions.

   Non-zero only over a small part of the domain (compact support)




$\Phi_1$ & $\Phi_2$ are trivially ortlogonal

$\Phi_1, \Phi_2, \Phi_3$ are almost orthogonal

FEM,

Specific choice

$$\theta_N(x) = \sum_{i=0}^{N} \underset{\theta_i}{\underset{\uparrow}{C_i'}} \; \underset{\Phi_i(x)}{\underset{\uparrow}{x^i}} \qquad \text{Global basis}$$

Any sine or cosine expansion can be used

$$\theta_N(x) = C_0' + C_1' x + C_2' x^2 + \cdots$$

Ex) $R(x) = \frac{d}{dx}\left[(1+a\theta)\frac{d\theta}{dx}\right] = 0$

$\theta(0) = 0$ Essential B.C

$\theta(1) = 1$

$\downarrow$

$C_0' = 0, \quad \sum_{i=0}^{N} C_i' = 1$

Force the approximation to satisfy these BC

$$\boxed{\theta_N(x) = x + \sum_{j=1}^{1} C_j (x^{j+1} - x)}$$

Automatically satisfies BC

Form residual

$$R(x; \theta_i) = \frac{d}{dx}\left[(1+a\theta_N)\frac{d\theta_N}{dx}\right]$$

Form Weighted Residual Eqn

$$\int_0^1 W_k(x)\, R(x; \theta_i)\, dx = 0 \qquad k = 1 \cdots N$$

1) Collocation

$W_k(x) = \delta(x - x_k)$ $\quad 0 \le x_k \le 1$

collocation pts

$\Rightarrow R(x_k; c_i) = 0$ — $R(x; c_i)$ is satisfied at a point $x_k$

<u>Question</u>: Converges with increasing N? yes

Does it depend on the choice of $\{x_k\}$ ~ very much.

2) Method of Moments.

$W_k(x) = x^{k-1}$

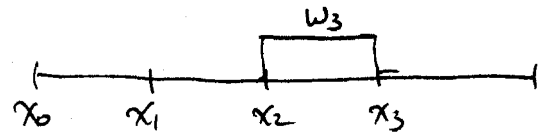
Virial methods.

3) Subdomain method (von-Karman methods)

not so useful

$W_k(x) = 1$ if $x_{k-1} < x < x_k$

$\quad\quad\quad\; 0$ elsewhere




4) Galerkin method

$W_k(x) = \Phi^k(x)$

Choose basis fn for weighting fn

Collocation, Galerkin } convergence is proven

5) Least Squares.

$$I = \int_0^1 R^2(x; c_i)\,dx$$ A functional

$I > 0$ always for an approx soln.

Minimize $I(\underline{c})$, $\underline{c}^T = (c_1, c_2 \ldots c_N)$

$$\Rightarrow \frac{\partial I}{\partial c_i} = 0 = \int_0^1 \cancel{2} R(x; c_i) \underbrace{\left[\frac{\partial R}{\partial c_k}\right]}_{\text{weighting fn}} dx = 0$$

$$w^k(x) = \cancel{2}\left(\frac{\partial R}{\partial c_k}\right)$$

6) Petrov – Galerkin

$$W_k(x) = \underbrace{\Phi^k(x)}_{\text{Basis Fn}} + \underbrace{N^k(x)}_{\text{Extra piece}}$$

*

Eg)

$$\frac{d}{dx}\left[(1+a\theta)\frac{d\theta}{dx}\right]=0$$

$$\theta_N(x) = x + \sum_{j=1}^{N} C_j (x^{j+1}-x)$$

$N=1, \quad a=1$

$$\theta_1 = x + C_1(x^2-x)$$

$$\theta_1' = 1 + C_1(2x-1)$$

$$\theta_1'' = 2C_1$$

$$R(x) = [1+x+C_1(x^2-x)]\,2C_1 + [1+C_1(2x-1)]^2 = 0$$

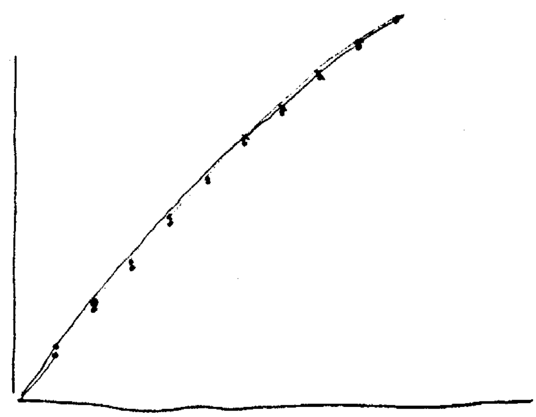
Collocation pick $x_1 = \frac{1}{2}$

$$R(x_1=\tfrac{1}{2};C_1) = -\tfrac{1}{2}C_1^2 + 3C_1 + 1 = 0$$

$$C_1 = 3 \pm \sqrt{11}$$

$$= -0.317 \quad \text{or} \quad 6.317$$

Exact solu : $\theta = -1 + (1+3x)^{1/2}$



$$\left.\frac{d\theta}{dx}\right|_{x=1} = 1 + c_1(2-1) = 1 + c_1 = \begin{matrix} 0.683 & \text{close} \\ 7.317 & \text{garbage} \end{matrix}$$

$$\left.\frac{d\theta_{ex}}{dx}\right|_{x=1} = \left.\frac{1}{2}\left((1+3x)^{-\frac{1}{2}} \cdot 3\right)\right|_{x=1} = \frac{3}{4}$$

$$\left.\frac{d\theta}{dx}\right|_{x=0} = 1 \cancel{+} - c_1 = \begin{matrix} 1.317 & \text{close} \\ -5.317 & \text{garbage} \end{matrix}$$

$$\left.\frac{d\theta}{dx}\right|_{x=0} = \frac{3}{2}$$

Take $N=2$

$$\theta_2 = x + c_1(x^2 - x) + c_2(x^3 - x)$$

2 unknowns $c_1$ & $c_2$

$$\theta_2' = 1 + c_1(2x-1) + c_2(3x^2-1)$$

$$\theta_2'' = 2c_1 + 6c_2 x$$

$$R(x_j; c_1, c_2) = [1 + x + c_1(x^2-x) + c_2(x^3-x)][2c_1 + 6c_2 x] + [1 + c_1(2x-1) + c_2(3x^2-1)]^2 = 0$$

pick $x_1 = \frac{1}{3}$, $x_2 = \frac{2}{3}$

2 eqns

$$a_{11}c_1^2 + a_{12}c_2^2 + a_{13}c_1c_2 + a_{14}c_1 + a_{15}c_2 + a_{16} = 0$$

$$a_{21}c_1^2 + a_{22}c_2^2 + a_{23}c_1c_2 + a_{24}c_1 + a_{25}c_2 + a_{26} = 0$$

$\Longrightarrow$ 4 solns

$$\left.\begin{matrix} c_1 = -0.5992 \\ c_2 = 0.1916 \end{matrix}\right)$$

## Galerkin's method

$N = 1$

$$\theta_1 = x + c_1 \underbrace{(x-1)x}_{\Phi^1(x)}$$

$$\int_0^1 \underbrace{(x-1)x}_{\Phi^1(x)} \; R(x; c)\, dx = 0$$

1 quad non-linear eq

$\rightarrow c_1 = -0.326.$

## Least-Squares

$$\int_0^1 \frac{\partial R}{\partial c_1} R(x; c_1)\, dx = 0$$

# Orthogonal Collocation

Let's assume the soln written as

$$y(x) = \sum_{i=0}^{N+1} a_i P_i(x)$$

orthogonal polynomial

If $y(0) = 0$ & $y(1) = 1$

more convenient form is

$$y = x + x(1-x) \sum_{i=1}^{N} a_i P_{i-1}(x)$$

where

$$P_{i-1}(x) = \sum_{j=0}^{i-1} c_j x^j$$

$$\Rightarrow \quad y = x + x(1-x) \sum_{i=1}^{N} a_i P_{i-1}(x)$$

$$= \sum_{i=1}^{N+2} b_i P_{i-1}(x)$$

$$= \sum_{i=1}^{N+2} d_i x^{i-1}$$

Transformation Matrices

$$\alpha_j \equiv y(x_j) = \sum_{i=1}^{N+2} d_i x_j^{i-1}$$

$$\rightarrow \quad \underline{y}^T = (y(x_1), y(x_2) \cdots ) = (\alpha_1, \alpha_2, \cdots )$$

$$\underline{y} = \underline{\underline{Q}}\, \underline{d}$$

$$Q_{ij} = x_j^{i-1}$$

$$* \quad \frac{d\underline{y}}{dx} = \underline{\underline{C}}\,\underline{d}$$

$$C_{ij} = (i-1)\,x_j^{\,i-2}$$

$$\underline{d} = \underline{\underline{Q}}^{-1}\underline{y}$$

$$\frac{d\underline{y}}{dx} = [\,\underline{\underline{C}}\,\underline{\underline{Q}}^{-1}\,]\,\underline{y}$$

$$= \underline{\underline{A}}\,\underline{y} \quad : \quad \underline{\underline{A}} = \underline{\underline{C}}\,\underline{\underline{Q}}^{-1}$$

$$\frac{d^2\underline{y}}{dx^2} = \underline{\underline{D}}\,\underline{d}$$

$$D_{ij} = (i-1)(i-2)\,x_j^{\,i-3}$$

$$= \underline{\underline{B}}\,\underline{y} \quad : \quad \underline{\underline{B}} = \underline{\underline{D}}\,\underline{\underline{Q}}^{-1}$$

We should know what collocation points are used.

Select collocation pt as

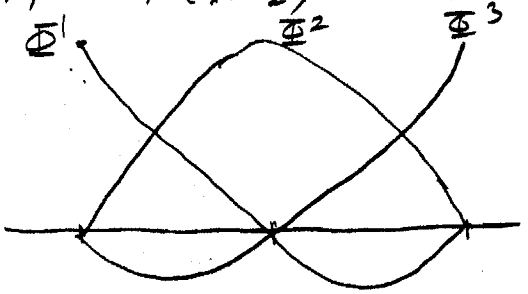
$\{x_i\}$ being the N roots of $P_N(x)$

Pick a basis $\quad 0 \le x \le 1$

$$\Phi^1(x) = 2(x-1)(x-\tfrac{1}{2})$$

$$\Phi^2(x) = 4x(1-x)$$

$$\Phi^3(x) = 2x(x-\tfrac{1}{2})$$




If $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{2} \\ 1 \end{bmatrix} \rightarrow \Phi^i(x_j) = \delta_{ij}$

$$\underline{\underline{M}} = \underline{\underline{I}} \rightarrow \alpha_i = a_i$$

## Orthogonal Polynomials

$P_m(x) = \sum_{j=0}^{m} c_j x^j \qquad$ degree $m$ : highest exponent

order $m+1$ : # of coeff

Condition for orthogonality (orthonormal)

$$\int_a^b W(x)\, P_i(x)\, P_j(x)\, dx = \delta_{ij}$$

$$\langle P_i(x), P_j(x) \rangle = \delta_{ij}$$

To construct sequences of orthogonal polynomial, must choose two things

$W(x)$ & $[a, b]$

# Legendre Polynomials

$w(x) = 1, \quad [a,b] = [-1,1]$

$P_0(x) = 1$

$P_1(x) = x$

$P_2(x) = \frac{1}{2}(3x^2-1)$

$P_3(x) = \frac{1}{2}(5x^3-3x)$

$\int_{-1}^{1} P_m(x) P_n(x)\, dx = 0$

$\int_{-1}^{1} \left(P_n(x)\right)^2 dx = \frac{2}{2n+1}$

analogy $\quad x = \cos\theta$

$\frac{d\theta}{dx} = -\sin\theta$

$$\int_{-1}^{1} P_m(x)\, P_n(x)\, dx \rightarrow \int_0^{\pi} P_m(\cos\theta)\, P_n(\cos\theta) \sin\theta\, d\theta$$

# Hermite Polynomials

$w(x) = e^{-x^2}, \quad [a,b] = [-\infty, \infty)$

$$\int_{-\infty}^{\infty} e^{-x^2} H_i(x) H_j(x)\, dx = \delta_{ij}$$

$H_0(x) = 1$

$H_1(x) = 2x$

$H_2(x) = 4x^2 - 2$

$H_3(x) = 8x^3 - 12x$

$\int_{-\infty}^{\infty} e^{-x^2} H_m(x) H_n(x) = 0$

$m \neq n$

$\int_{-\infty}^{\infty} e^{-x^2} \left(H_n(x)\right)^2 dx$

$= 2^n n! \sqrt{\pi}$

Do a problem

$$(1+a\theta)\frac{d^2\theta}{dx^2} + a\left(\frac{d\theta}{dx}\right)^2 = 0$$

$$\theta(0)=0, \quad \theta(1)=1$$

let $\theta(x) = \sum_{i=1}^{N+2} d_i x^{i-1}$

Collocation eqn by replacing $d_i$ with $\theta_i$

$$(1+a\theta_i)\sum_{j=1}^{N+2} B_{ij}\theta_j + a\left(\sum_{j=1}^{N+2} A_{ij}\theta_j\right)^2 = 0$$

$$i = 2, \cdots N+1$$

$$\begin{pmatrix} \theta_1 = 0 \\ \theta_{N+2} = 1 \end{pmatrix}, \quad \begin{pmatrix} x_1 = 0 \\ x_{N+2} = 1 \end{pmatrix}$$

$N+2$ eqn in $\{\theta_i\}$

$$\Rightarrow \quad \underline{R}(\underline{\theta}) = \underline{0}$$

Set $a=0$

$$\sum_{j=1}^{N+2} B_{ij}\theta_j = 0 \quad \Rightarrow \quad \underline{\underline{B}}\,\underline{\theta} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$\theta_1 = 0$$
$$\theta_{N+2} = 1$$

How sparse is $\underline{\underline{B}}$? dense matrix

check the effect of orthogonal polynomial.