Nonlinear Systems Analysis
I. Phase-Plane Analysis

Objectives:

- Use eigenvalues and eigenvectors of the Jacobian matrix to characterize the phase-plane behavior.
- Predict the phase-plane behavior close to an equilibrium point, based on the linearized model at that equilibrium point.
- Predict qualitatively the phase-plane behavior of the nonlinear system, when there are multiple equilibrium points.

Phase-plane analysis: one of most important techniques for studying the behavior of nonlinear systems.

Eigenvalue/eigenvector analysis: predict fast and slow (or stable and unstable) initial conditions.

Eigenvalue: speed of response / Eigenvector: principal direction

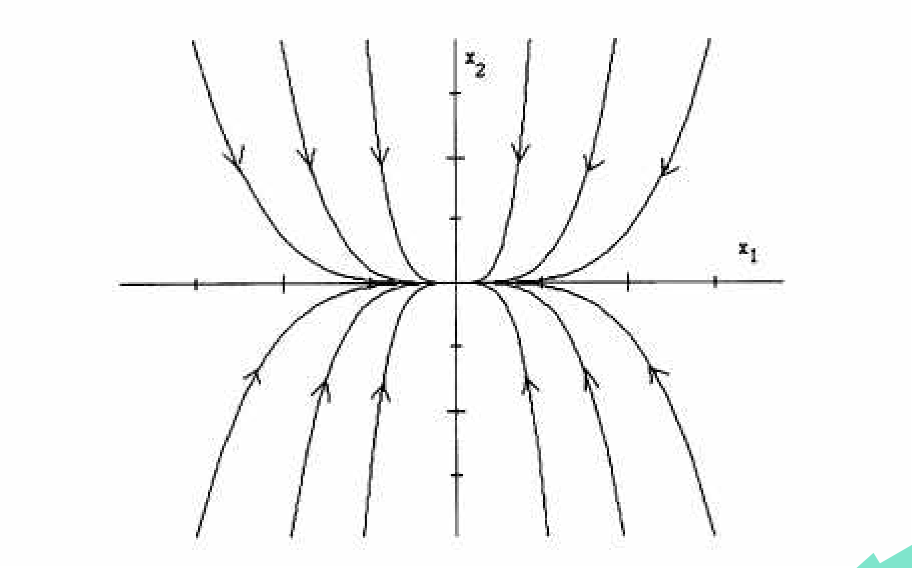
1. Linear System Examples

Ex.1) A stable equilibrium point (**node sink**):

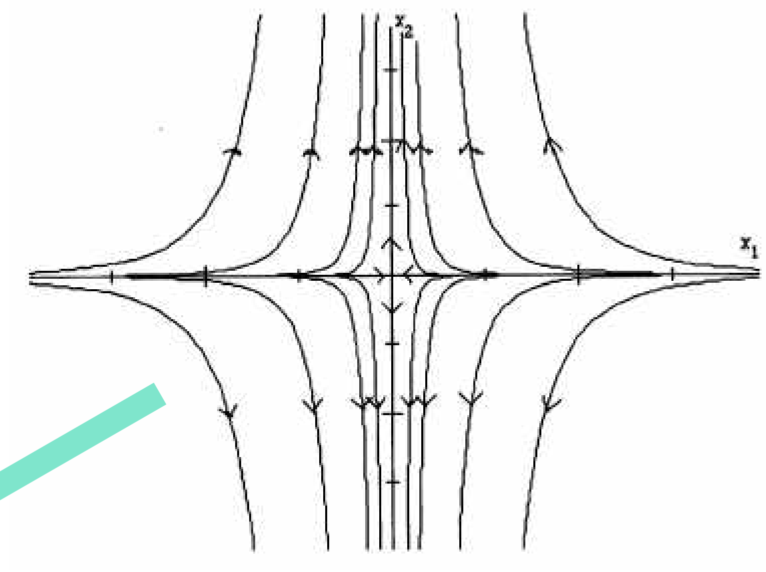
$\dot{x}_1 = -x_1$ Steady state: (0,0), Jacobian $\begin{pmatrix} -1 & 0 \\ 0 & -4 \end{pmatrix}$

$\dot{x}_2 = -4x_2$ Eigenvalues: -1, -4, Eigenvectors: $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ for $\lambda = -1$, $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ for $\lambda = -4$

Phase-plane for the node sink problem



Phase-plane for the saddle problem



Ex.2) An unstable equilibrium point (**saddle**):

$$\dot{x}_1 = -x_1$$
$$\dot{x}_2 = 4x_2$$

Steady state: (0,0), Jacobian $\begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix}$

Eigenvalues: -1, 4, Eigenvectors: $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ for $\lambda = -1$, $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ for $\lambda = 4$

Separatrix: a line in the phase-plane that is not crossed by any trajectory.
Eigenvectors are the separatrices in the general case.
In the above example, separatrices are the coordinates axes.

Ex.3) Another saddle problem

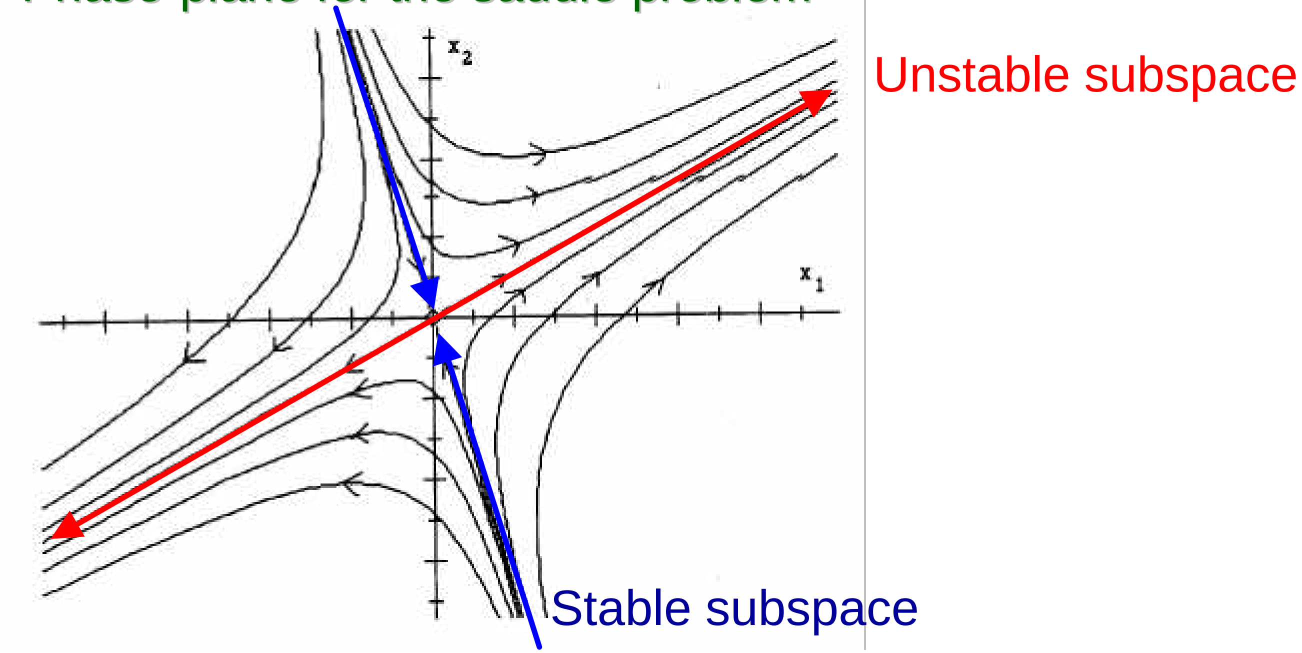
$$\dot{x}_1 = -2x_1 + x_2$$
$$\dot{x}_2 = 2x_1 - x_2$$

Steady state: (0,0), Jacobian $\begin{pmatrix} 2 & 1 \\ 2 & -1 \end{pmatrix}$

Eigenvalues: $\lambda_1 = -1.5616, \lambda_2 = 2.5616$

Eigenvectors: $\begin{pmatrix} 0.2703 \\ -0.9628 \end{pmatrix}$ for λ_1, $\begin{pmatrix} 0.8719 \\ 0.4896 \end{pmatrix}$ for λ_2

Phase-plane for the saddle problem



Ex.4) Unstable focus (**spiral source**)

$$\dot{x}_1 = x_1 + 2x_2$$
$$\dot{x}_2 = -2x_1 + x_2$$

Steady state: (0,0), Jacobian $\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$

Eigenvalues: $\lambda_1 = 1 + 2j, \lambda_2 = 1 - 2j$

(positive real part of eigenvalue → unstable)

Ex.5) Center

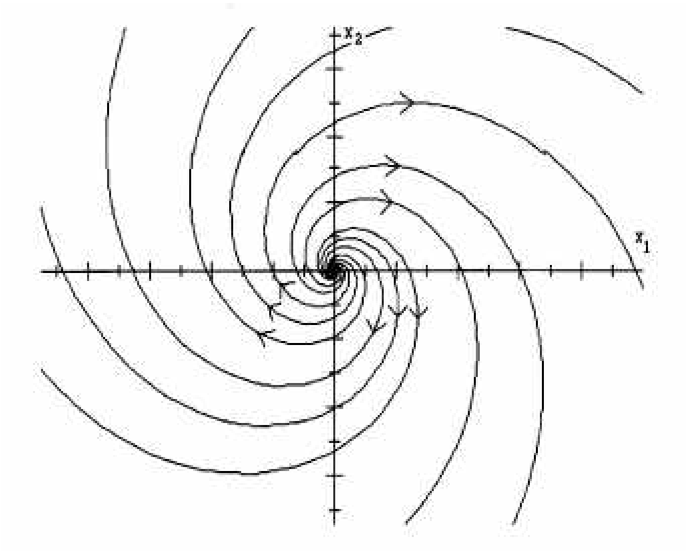
$$\dot{x}_1 = -x_1 - x_2$$
$$\dot{x}_2 = 4x_1 + x_2$$

Steady state: (0,0), Jacobian $\begin{pmatrix} -1 & -1 \\ 4 & 1 \end{pmatrix}$

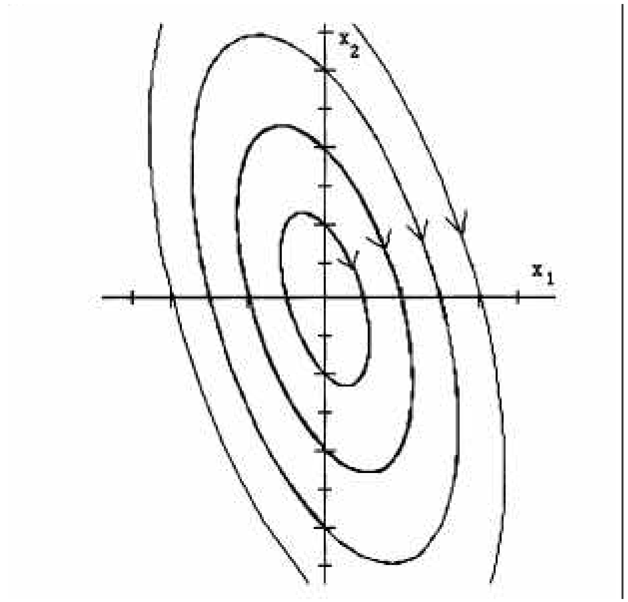
Eigenvalues: $\lambda_1 = 0 + 1.7321j, \lambda_2 = 0 - 1.7321j$

(sinusoidal periodic solution)

Phase-plane for the unstable focus problem

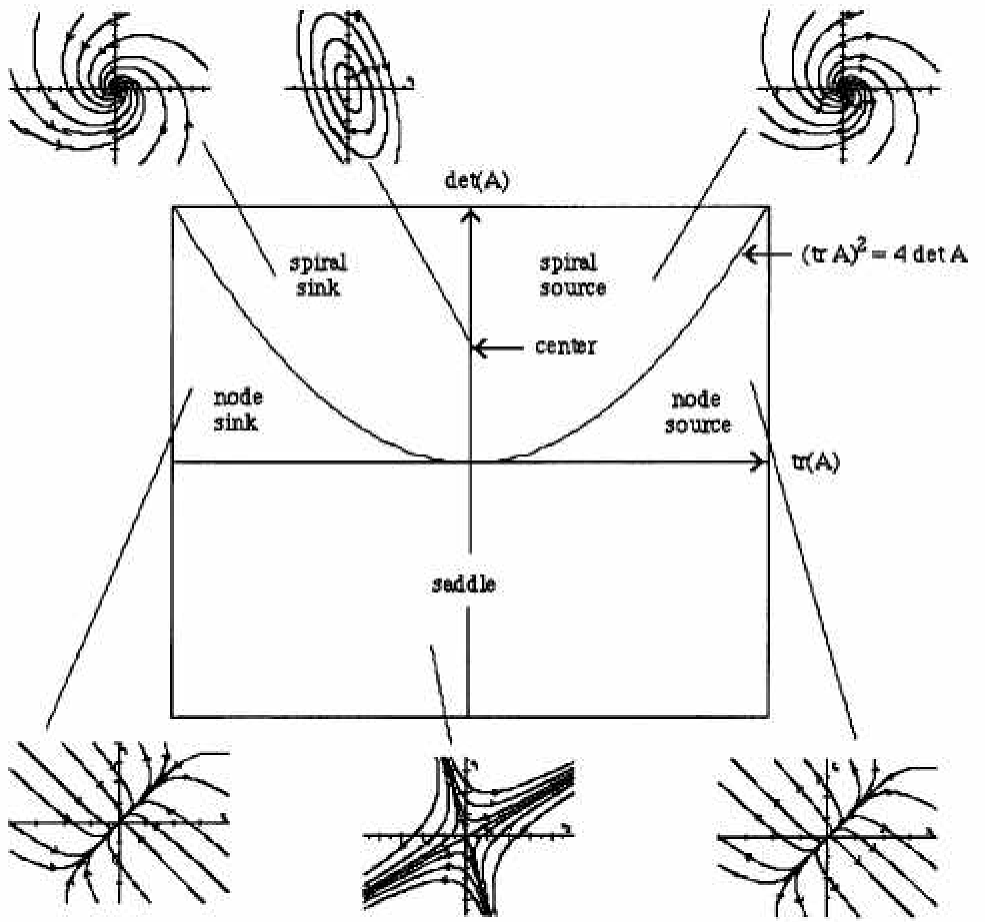


Phase-plane for the center problem



2. Generalization of Phase-plane Behavior

$$\dot{\underline{x}} = \underline{\underline{A}}\,\underline{x} \quad \text{where } \underline{\underline{A}} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$



Eigenvalues:

$$\lambda = \frac{\mathrm{tr}(A) \pm \sqrt{(\mathrm{tr}(A))^2 - 4\det(A)}}{2}$$

Autonomous system:
trajectories cannot cross in the plane.
Non-autonomous system:
trajectories could cross…

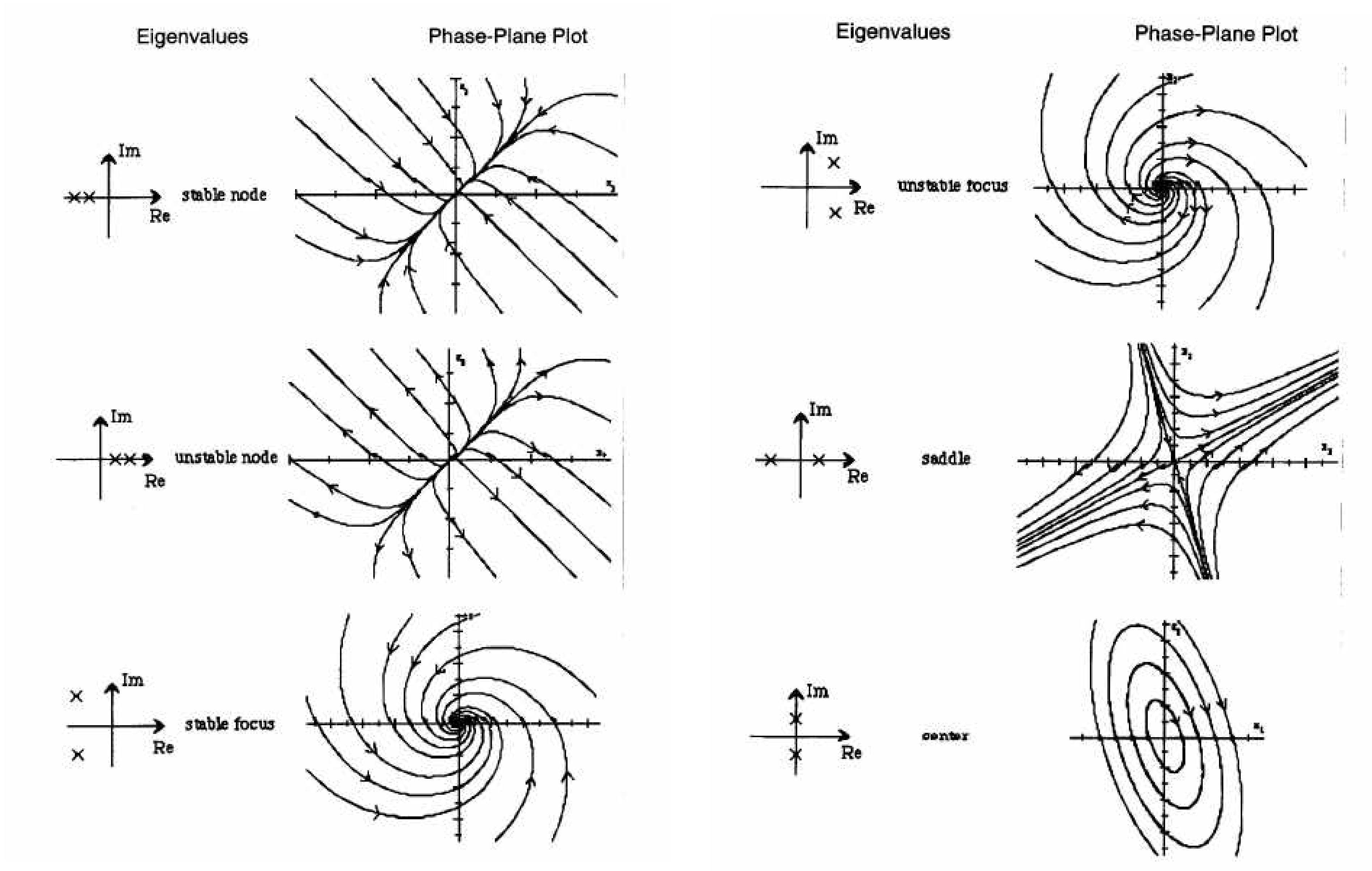
Sinks (stable nodes): $\mathrm{Re}(\lambda_1)<0$ and $\mathrm{Re}(\lambda_1)<0$

Saddles (unstable): $\mathrm{Re}(\lambda_1)<0$ and $\mathrm{Re}(\lambda_2)>0$

Sources (unstable nodes): $\mathrm{Re}(\lambda_1)>0$ and $\mathrm{Re}(\lambda_2)>0$

Spirals: λ_1 and λ_2 are complex conjugates. If $\mathrm{Re}(\lambda_1)<0$ then stable,
if $\mathrm{Re}(\lambda_1)>0$ then unstable.

Generalization of phase-plane behavior:



3. Nonlinear Systems

Ex. 6) Nonlinear (Bilinear) system

$$\frac{dz_1}{dt} = z_2(z_1 + 1)$$
$$\frac{dz_2}{dt} = z_1(z_2 + 3)$$

Steady states: trivial: $z_{1s} = 0, z_{2s} = 0$

nontrivial: $z_{1s} = -1, z_{2s} = -3$

Jacobian matrix: $\underline{\underline{J}} = \begin{bmatrix} z_{2s} & z_{1s} + 1 \\ z_{2s} + 3 & z_{1s} \end{bmatrix}$

Case 1) Steady state 1 (Trivial)

Eigenvalues: $\lambda_1 = -\sqrt{3}, \lambda_2 = \sqrt{3}$

Eigenvectors: $\begin{pmatrix} -0.5 \\ 0.866 \end{pmatrix}$ for λ_1, $\begin{pmatrix} 0.5 \\ 0.866 \end{pmatrix}$ for λ_2

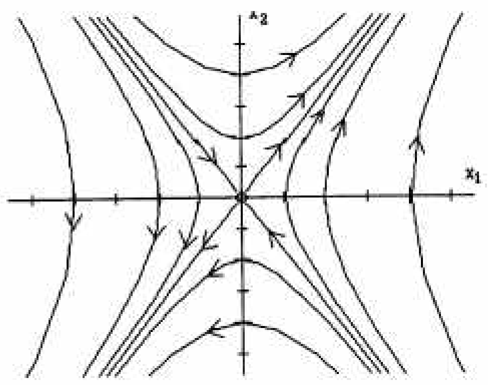
Case 2) Steady state 2 (Nontrivial)

Eigenvalues: $\lambda_1 = -3, \lambda_2 = -1$

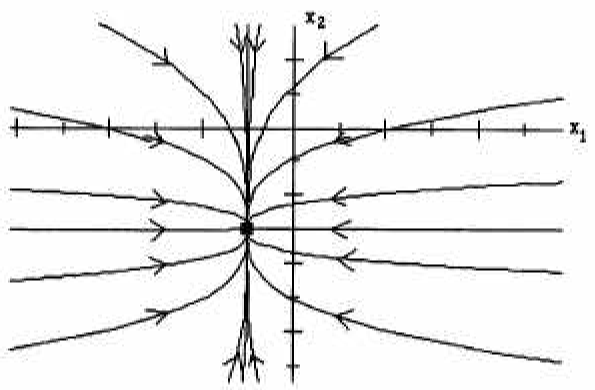
Eigenvectors: $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ for λ_1, $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ for λ_2

fast stable *slow stable*

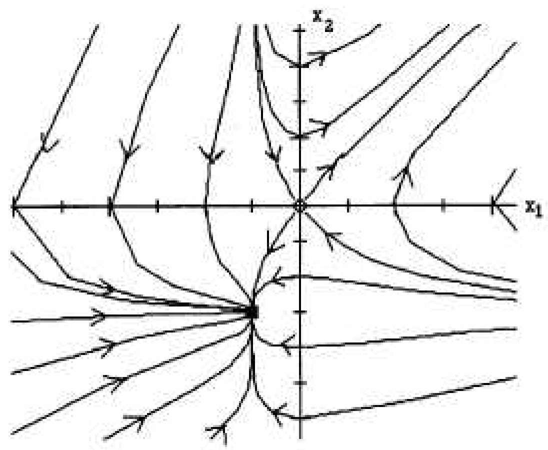
Phase-plane around trivial point



Phase-plane around nontrivial point



Phase-plane of nonlinear system



Ex. 7) Bioreactor with Monod kinetics

$$\frac{dx_1}{dt} = (\mu - D)x_1$$
$$\frac{dx_2}{dt} = (s_f - x_2)D - \frac{\mu x_1}{Y}$$

$$\mu = \frac{\mu_{max} x_2}{k_m + x_2},$$
$$\mu_{max} = 0.53,\ k_m = 0.12,\ Y = 0.4,\ s_f = 4.0$$

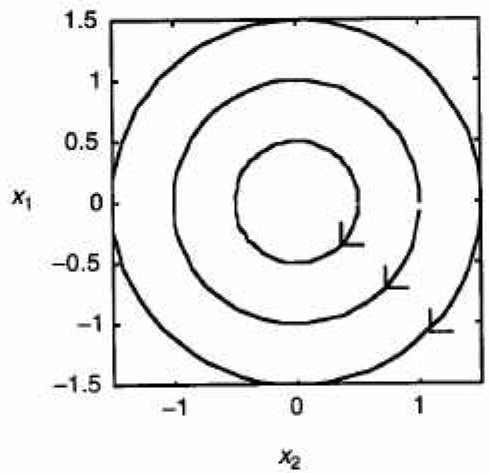
Determine the stability of the steady states and plot the phase-plane around the steady states.

Difference between center and limit cycle

- Center trajectories can be found in the linear or linearized systems with the largest real part of the eigenvalues of zero value (in marginal point.)
  → depend on the initial conditions

- Limit cycle can occur in nonlinear systems:
  → isolated closed orbit
  (related to Hopf bifurcation)

Center



Limit cycle