Nonlinear Systems Analysis
III-1. Bifurcation Behavior of Single ODE systems

Objectives:

- Determine the bifurcation point for a single ODE
- Determine the stability of each branch of a bifurcation diagram
- Determine the number of steady-state solutions near a bifurcation point

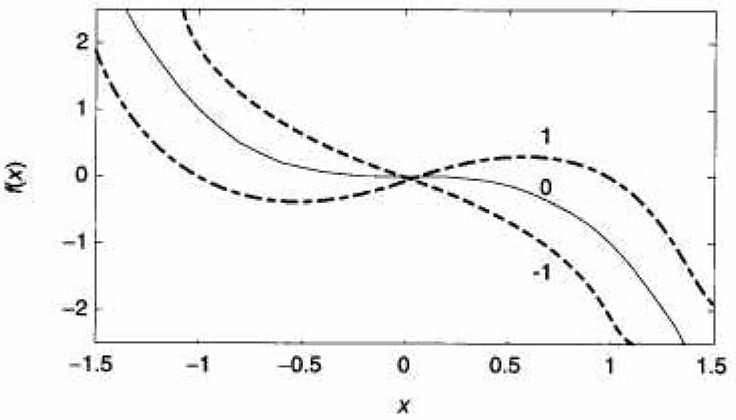
Bifurcation occurs if the number of steady-state solutions changes as a system parameter is changed. If the qualitative (stable vs unstable) behavior of a system changes as a function of a parameter, we also refer to this as bifurcation behavior.

- Important for complex systems such as chemical and biochemical reactors.

1. Illustration of Bifurcation Behavior

$$f(x,\mu) = \mu x - x^3 = 0$$

$\mu<0$: one steady-state
$\mu>0$: three steady-states
→ $\mu=0$: bifurcation point
(pitchfork bifurcation)



2. Types of Bifurcation

- Pitchfork bifurcation
- Saddle-node bifurcation
- Transcritical bifurcation

- Consider general dynamic equation: $\dot{x} = f(x,\mu)$ steady-state: $0 = f(x,\mu)$

Bifurcation point: $f(x,\mu) = \frac{\partial f}{\partial x} = 0$ (first derivative: Jacobian for the single Eqn.)

→ eigenvalue = 0 at a bifurcation point

Number of solutions from catastrophe theory:

$$f(x,\mu) = \frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial x^2} = \ldots = \frac{\partial^{k-1} f}{\partial x^{k-1}} = 0 \quad \text{and} \quad \frac{\partial^k f}{\partial x^k} \neq 0$$

Example 1: **Pitchfork Bifurcation**

$\dot{x} = f(x,\mu) = \mu x - x^3$ Steady state solutions: $x_{s0} = 0,\ x_{s1} = \sqrt{\mu},\ x_{s2} = -\sqrt{\mu}$

Jacobian: $\left.\frac{\partial f}{\partial x}\right|_{x_s,\mu_s} = -3x_s^2 + \mu_s = \lambda(\text{eigenvalue})$

(1) $\mu<0$: one steady-state, $x_{s0}=0$, stability: stable (negative eigenvalue)

(2) μ>0: three steady-states

a. $x_{s0} = 0$: eigenvalue, $\lambda = \mu_s \rightarrow$ unstable

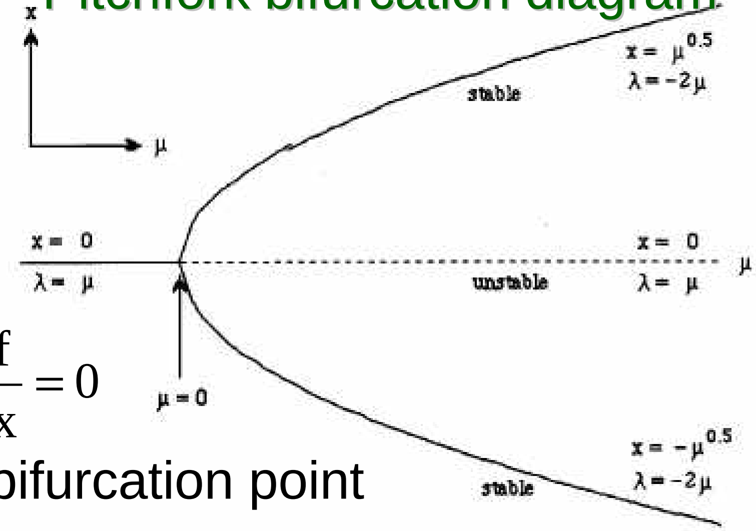
b. $x_{s1} = \sqrt{\mu}$: $\lambda = -2\mu_s \rightarrow$ stable

c. $x_{s2} = -\sqrt{\mu}$: $\lambda = -2\mu_s \rightarrow$ stable

Bifurcation point: $\mu_s = 0, x_s = 0$ from $f(x,\mu) = \frac{\partial f}{\partial x} = 0$

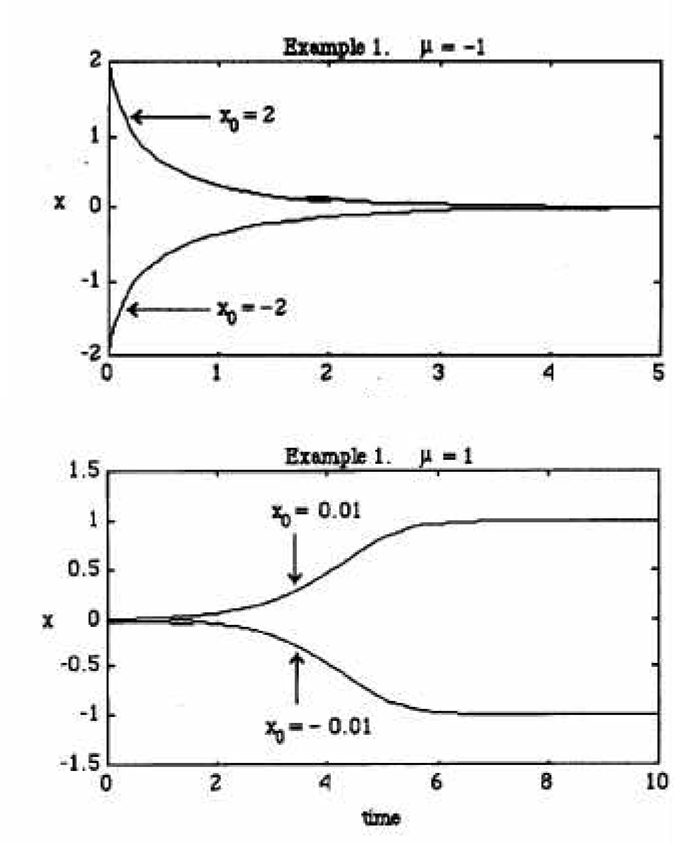
Number of solutions: **3** in the vicinity of the bifurcation point

Pitchfork bifurcation diagram

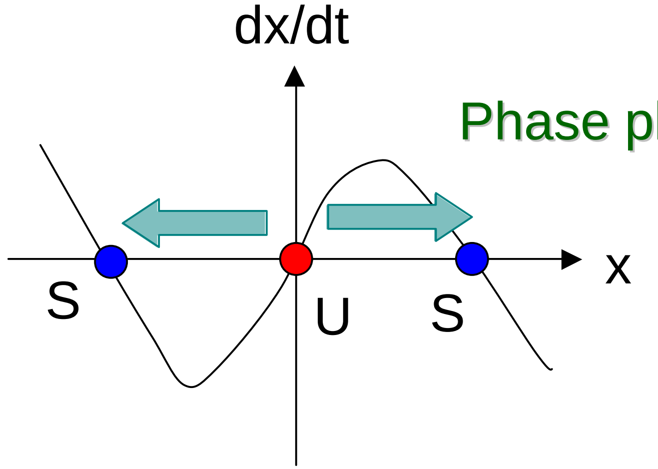


Dynamic response

$$\left(\left. \frac{\partial^3 f}{\partial x^3} \right|_{x_s, \mu_s} \neq 0 \right)$$



Phase plane



Example 2: Saddle-Node Bifurcation (Turning Point)

$\dot{x} = f(x,\mu) = \mu - x^2$ Steady state solutions: $x_{s1} = \sqrt{\mu},\ x_{s2} = -\sqrt{\mu}$

Jacobian: $\left.\frac{\partial f}{\partial x}\right|_{x_s,\mu_s} = -2x_s = \lambda(\text{eigenvalue})$

Bifurcation point: $\mu_s = 0, x_s = 0$

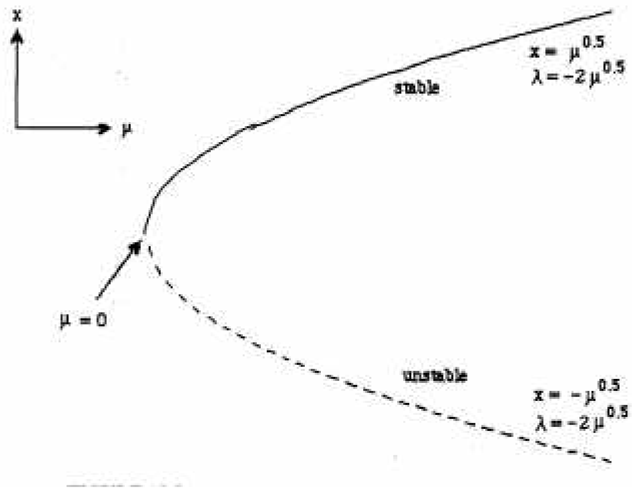
2 solutions around the bifurcation point $\left(\left.\frac{\partial^2 f}{\partial x^2}\right|_{x_s,\mu_s} \neq 0 \right)$

(1) $\mu<0$: no real solutions

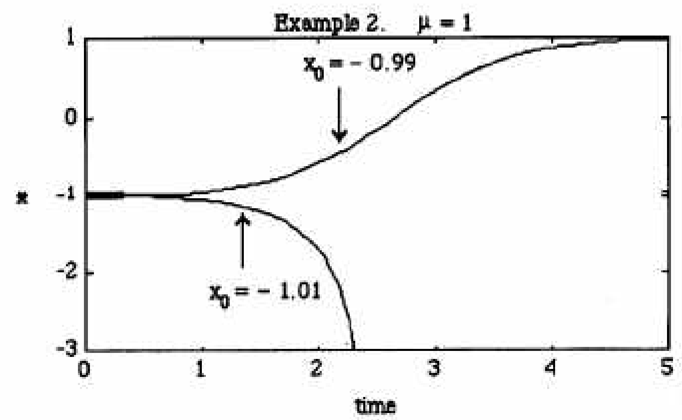
(2) $\mu>0$: a. $x_{s1} = \sqrt{\mu}:\ \lambda = -2\sqrt{\mu} \rightarrow \text{stable}$

b. $x_{s2} = -\sqrt{\mu}:\ \lambda = 2\sqrt{\mu} \rightarrow \text{unstable}$

Saddle-node bifurcation diagram



Dynamic response



Example 3: Transcritical Bifurcation

$\dot{x} = f(x,\mu) = \mu x - x^2$ Steady state solutions: $x_{s1} = 0,\ x_{s2} = \mu$

Jacobian: $\left.\frac{\partial f}{\partial x}\right|_{x_s,\mu_s} = \mu - 2x_s = \lambda(\text{eigenvalue})$

Bifurcation point: $\mu_s = 0, x_s = 0$

2 solutions around the bifurcation point $\left(\left.\frac{\partial^2 f}{\partial x^2}\right|_{x_s,\mu_s} \neq 0 \right)$

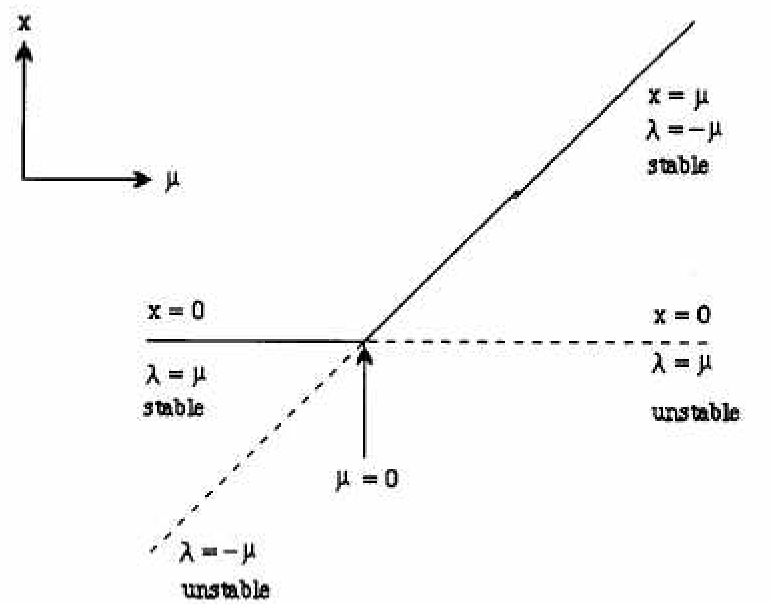
(1) μ<0: a. $x_{s1} = 0:\ \lambda = \mu \rightarrow \text{stable}$

b. $x_{s2} = \mu:\ \lambda = -\mu \rightarrow \text{unstable}$

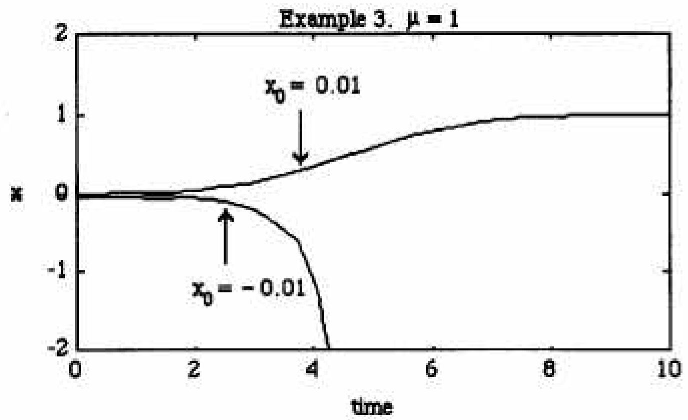
(2) μ>0: a. $x_{s1} = 0:\ \lambda = \mu \rightarrow \text{unstable}$

b. $x_{s2} = \mu:\ \lambda = -\mu \rightarrow \text{stable}$

Transcritical bifurcation diagram



Dynamic response



Example 4: Hysteresis Behavior

$$\dot{x} = f(x,\mu) = u + \mu x - x^3$$

u: adjustable input parameter
μ: design-related parameter

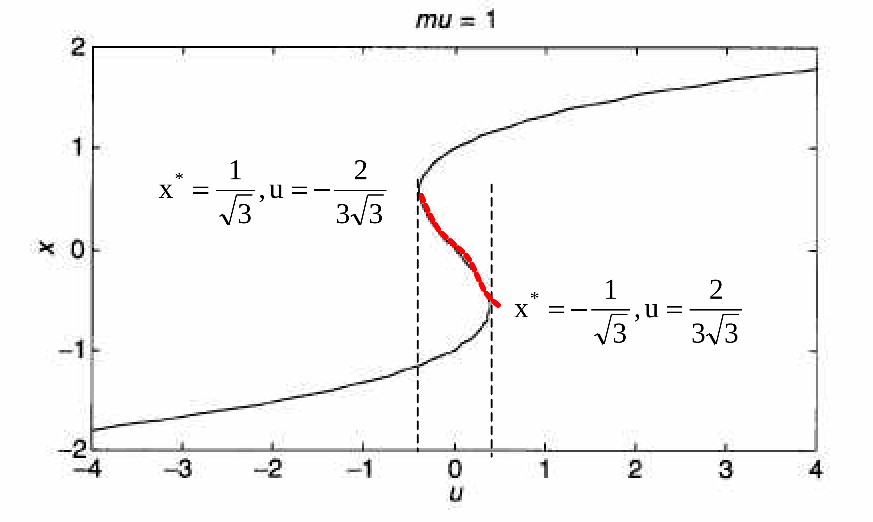
$$\left(\left.\frac{\partial^2 f}{\partial x^2}\right|_{x_s,\mu_s} \neq 0 \right)$$

(1) μ=-1: Steady state solutions: $u = x_s + x_s^3$

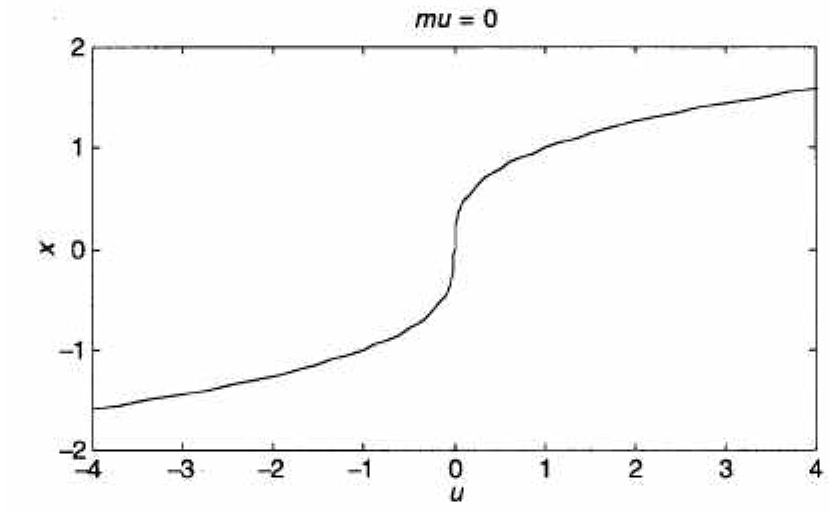
Jacobian: $\left.\frac{\partial f}{\partial x}\right|_{x_s,\mu_s} = -1 - 3x_s^2$ always negative (no bifurcation point)

(2) μ=1: Steady state solutions: $u + x_s - x_s^3 = 0$ for example:

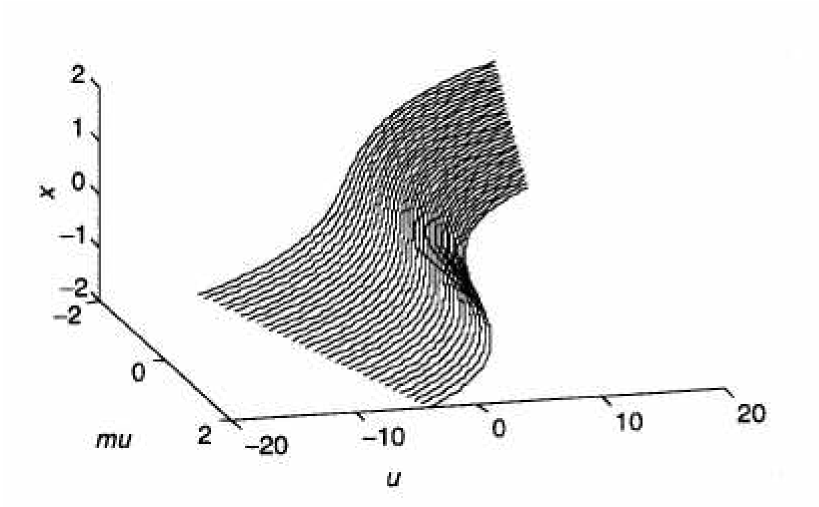
$$x_s = 1(s),\ 0(u),\ -1(s)\ \text{for } u = 0$$



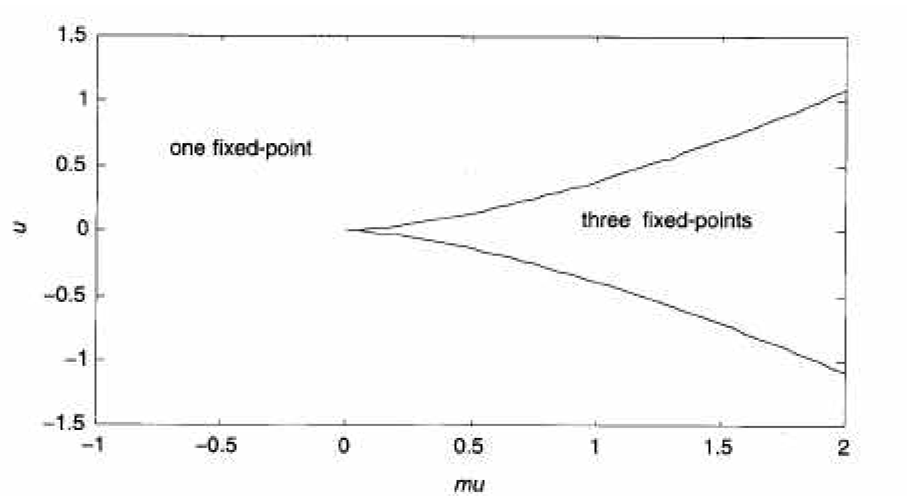
Hysteresis behavior



Cusp catastrophe diagram



Two-parameter bifurcation diagram



Nonlinear Systems Analysis
III-2. Bifurcation Behavior of Two-State Systems

Objectives:

- Find bifurcations that occur in two-state systems (pitchfork, saddle-node, transcritical)
- Understand the difference between limit cycles (nonlinear behavior) and centers (linear behavior)
- Distinguish between stable and unstable limit cycles
- Determine the conditions for a Hopf bifurcation (subcritical and supercritical)

1. Single Dimensional Bifurcation in the Phase-Plane

$$\dot{x}_1 = f_1(x,\mu) = \mu x_1 - x_1^3$$
$$\dot{x}_2 = f_2(x,\mu) = -x_2$$

Steady state solutions: $\underline{x}_s = (0,0) \text{ or } (\sqrt{\mu},0) \text{ or } (-\sqrt{\mu},0)$

Jacobian: $$\underline{\underline{J}} = \begin{bmatrix} \mu - 3x_{1s}^2 & 0 \\ 0 & -1 \end{bmatrix}$$

(1) $\mu<0$: one steady-state, $x_{s0}=0$, stability: stable

(2) $\mu=0$: one steady-state, $x_{s0}=0$, stability: stable

(3) $\mu>0$: three steady-states, $\underline{x}_s = (0,0) \text{ or } (\sqrt{\mu},0) \text{ or } (-\sqrt{\mu},0)$

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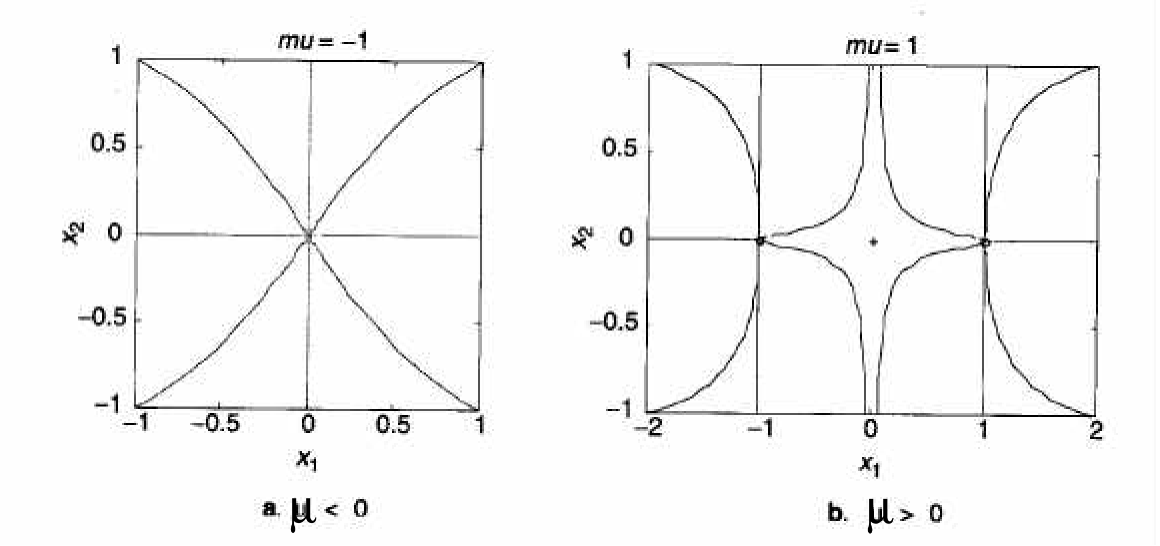
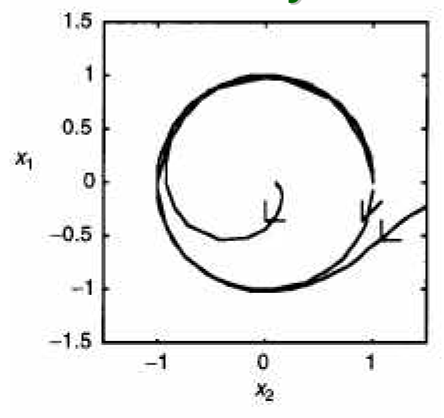
Pitchfork bifurcation behavior



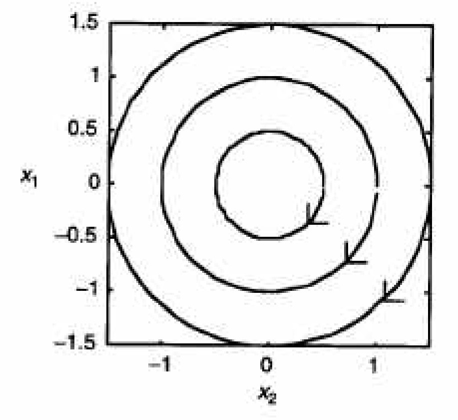
2. Limit Cycle Behavior

- Center occurs in linear systems that have eigenvalues with zero real part
  Different initial conditions → different closed-cycles.
- Limit cycles are isolated closed orbits in nonlinear systems.
  Perturbations in initial conditions
  → returns to the closed cycle
  (for stable limit cycle)

Limit cycle



Center



Ex) A Stable Limit Cycle

$$\dot{r} = r(1 - r^2)$$
$$\dot{\theta} = -1$$

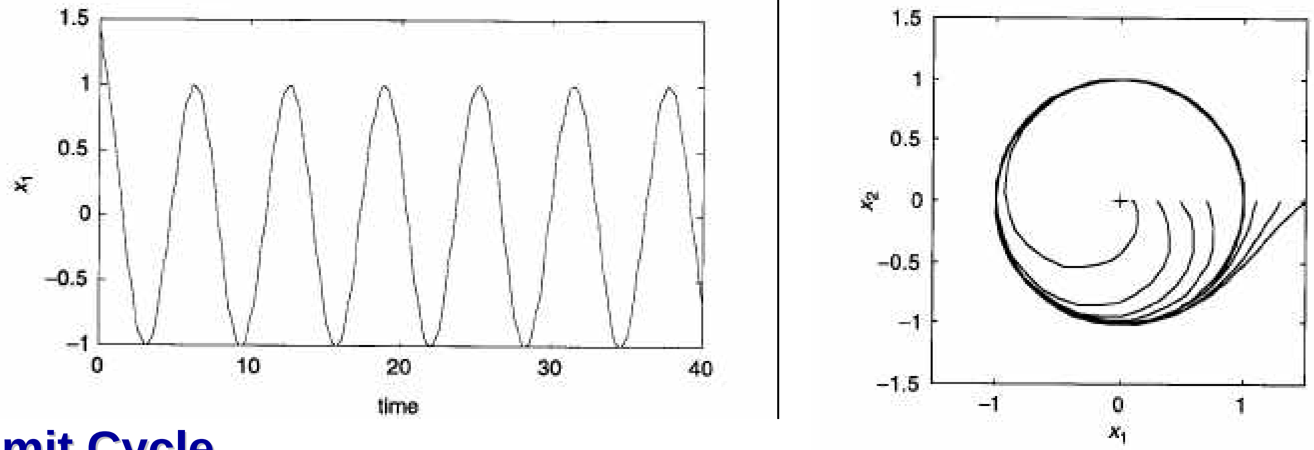
Steady state solutions: r=0 and r=1

Jacobian: $\frac{\partial f}{\partial x} = -1 - 3r^2$

→ r=0: unstable
r=1: stable

(Angle is constantly decreasing. Stability of this system is determined by the first eqn.)

Stable limit cycle behavior



Ex) An Unstable Limit Cycle

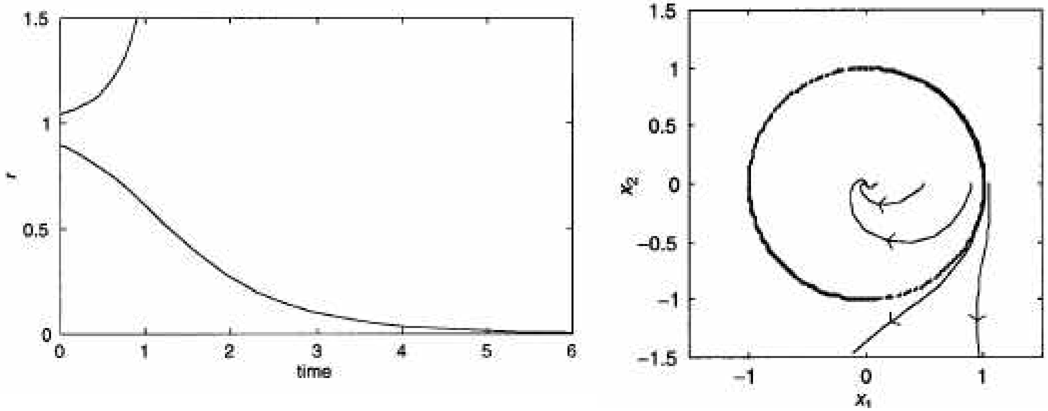
$$\dot{r} = -r(1 - r^2)$$
$$\dot{\theta} = -1$$

Steady state solutions: r=0 and r=1

Jacobian: $\frac{\partial f}{\partial x} = -1 + 3r^2$

→ r=0: stable
r=1: unstable

Unstable limit cycle behavior



3. Hopf Bifurcation

Remind: Point where the number of solutions changed was the bifurcation point.

An exchange of stability generally occurred at the bifurcation point.

Hopf bifurcation occurs when a limit cycle forms as a parameter is varied.

Ex) Supercritical Hopf Bifurcation

$$\dot{x}_1 = x_2 + x_1(\mu - x_1^2 - x_2^2)$$

$$\dot{x}_2 = -x_1 + x_2(\mu - x_1^2 - x_2^2)$$

$\Rightarrow$ in polar coordinates

$$\dot{r} = r(\mu - r^2),\ \ \dot{\theta} = -1$$

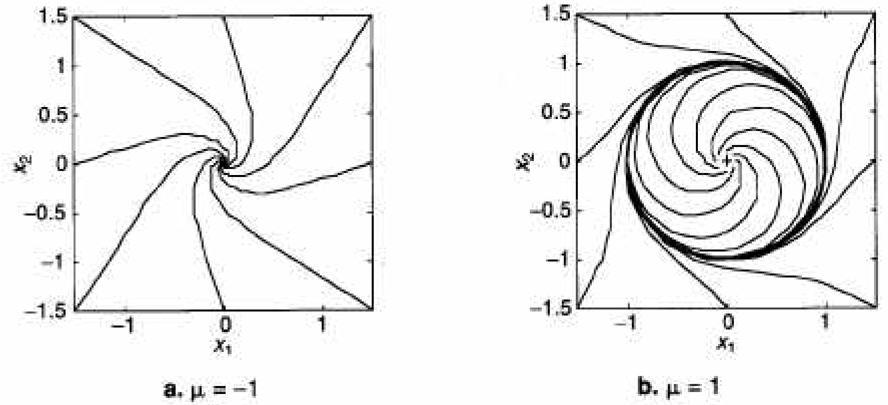
- Steady state solutions: $r = 0, \sqrt{\mu}, -\sqrt{\mu}$
- Jacobian: $\dfrac{\partial f}{\partial x} = \mu - 3r^2$
- Stability:

$(1)\ \mu < 0 : r = 0$ (stable)

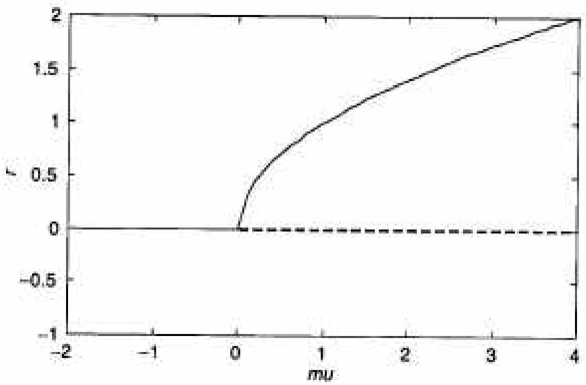
$(2)\ \mu = 0 : r = 0$ (stable)

$(3)\ \mu > 0 : r = 0$ (unstable), $\pm\sqrt{\mu}$ (stable)

Phase plane plot



Bifurcation diagram



- Determine the stability of this system in Cartesian coordinates

- Steady state: $\underline{x}_s = (0,0)$

- Jacobian:

$$\underline{\underline{J}} = \begin{bmatrix} \mu - 3x_1^2 - x_2^2 & 1 - 2x_1 x_2 \\ -1 - 2x_1 x_2 & \mu - x_1^2 - 3x_2^2 \end{bmatrix} \Rightarrow \begin{bmatrix} \mu & 1 \\ -1 & \mu \end{bmatrix}$$

- Eigenvalues: $\lambda = \mu \pm 1i$