Nonlinear Systems Analysis
IV. Introduction to Chaos: Lorenz Equations

Objectives:

- Understand what is meant by chaos (extreme sensitivity to initial conditions)
- Understand conceptually the physical system that the Lorenz equations attempt to model.
- Understand how the system behavior changes as the parameter r is varied.

Chaos occurring in continuous (ODE) models with three or more equations (autonomous model)

1. The Lorenz Equations

$$\dot{x}_1 = \sigma(x_2 - x_1)$$
$$\dot{x}_2 = rx_1 - x_2 - x_1x_3$$
$$\dot{x}_3 = -bx_3 + x_1x_2$$

(see the text for all nomenclatures)

(*r: ratio of the Rayleigh, Ra, to the critical Ra_c*)
r>1, convective flow; r<1, no convective flow

- Steady-state solutions

State variable	Trivial solution	Nontrivial 1 (r>1)	Nontrivial 2 (r>1)
x_{1s}	0	$\sqrt{b(r-1)}$	$-\sqrt{b(r-1)}$
x_{2s}	0	$\sqrt{b(r-1)}$	$-\sqrt{b(r-1)}$
x_{3s}	0	$r-1$	$r-1$

2. Stability Analysis of the Lorenz Equations

$$\text{Jacobian}: \underline{\underline{J}} = \begin{bmatrix} -\sigma & \sigma & 0 \\ r - x_{3s} & -1 & -x_{1s} \\ x_{2s} & x_{1s} & -b \end{bmatrix}$$

(1) Trivial solution:

$$\lambda_1 = -b$$

$$\lambda_2 = \frac{-(\sigma+1) - \sqrt{(\sigma+1)^2 - 4\sigma(1-r)}}{2}, \quad \lambda_2 = \frac{-(\sigma+1) + \sqrt{(\sigma+1)^2 - 4\sigma(1-r)}}{2}$$

r<1: all negative eigenvalues → stable
r>1: saddle point (one unstable eigenvalue) → unstable

(2) Nontrivial solutions: *(only exist for r>1)*

- Characteristic equation: $\lambda^3 + b_2\lambda^2 + b_1\lambda + b_0 = 0$

$$\text{where, } b_2 = \sigma + b + 1, \; b_1 = (r+\sigma)b, \; b_0 = 2\sigma b(r-1)$$

- Use Routh array for finding stability condition: *(see the stability of ODEs)*

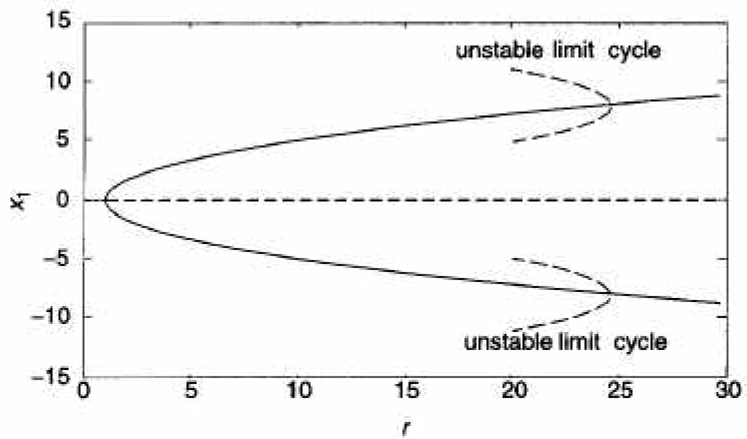
Critical r for stability:

$$r_H = \frac{\sigma(\sigma + b + 3)}{(\sigma - b - 1)}$$

(subscript H means Hopf bifurcation)

Supercritical Hopf …

(3) Summary of stability results



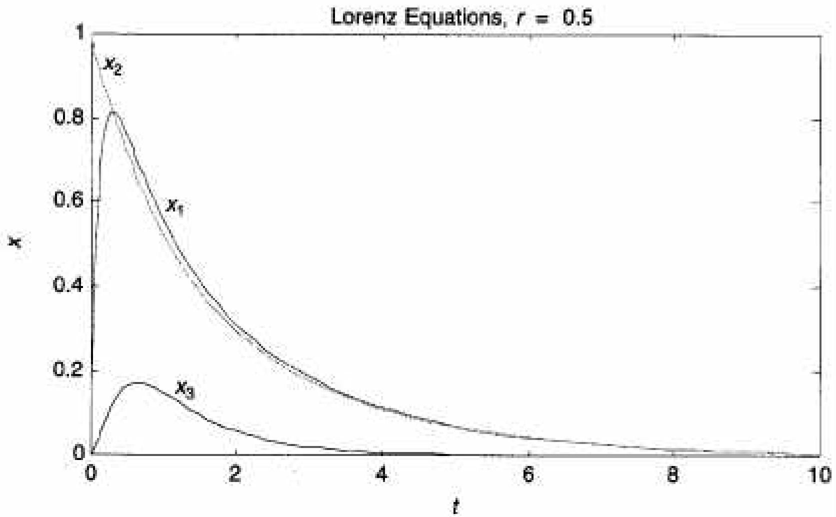
3. Numerical Study of the Lorenz Equations

(for σ=10, b=8/3)

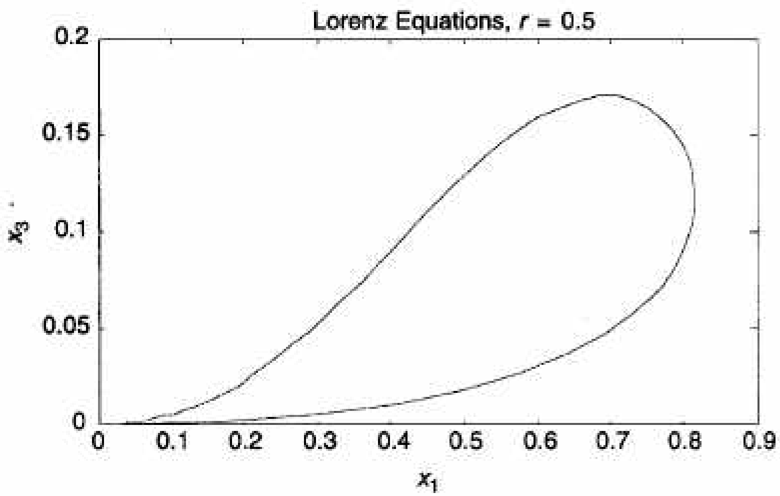
(1) For a stable trivial solution

(for σ=10, b=8/3, r=0.5; initial condition of x_0=(0,1,0))

Transient response



Phase plane (x_1 vs x_3)



(2) For a stable nontrivial solutions

(for σ=10, b=8/3, r=21)

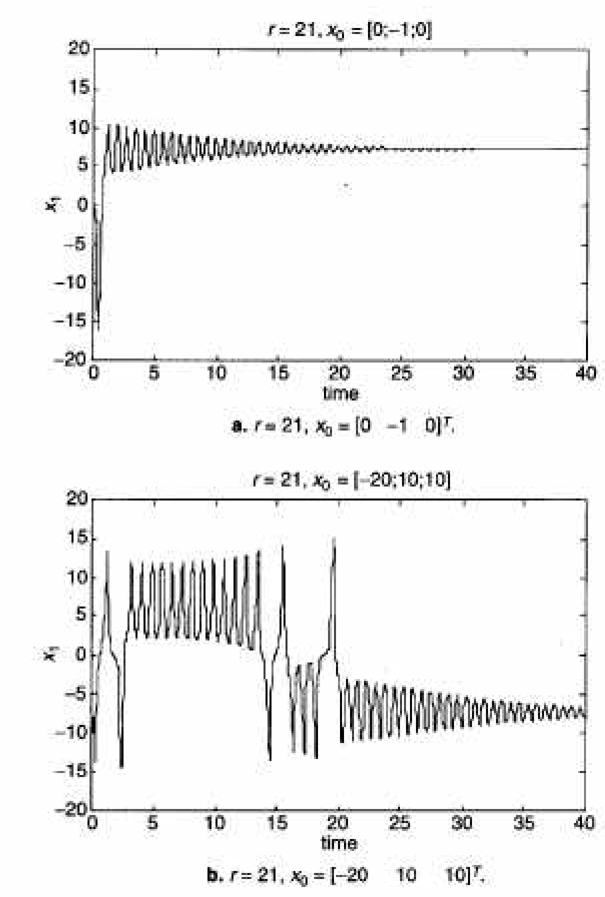
- *In this case, eigenvalues of trivial solution:* $\lambda_1 = -2.67; \lambda_2 = -20.67; \lambda_3 = 9.67$

  → *unstable*

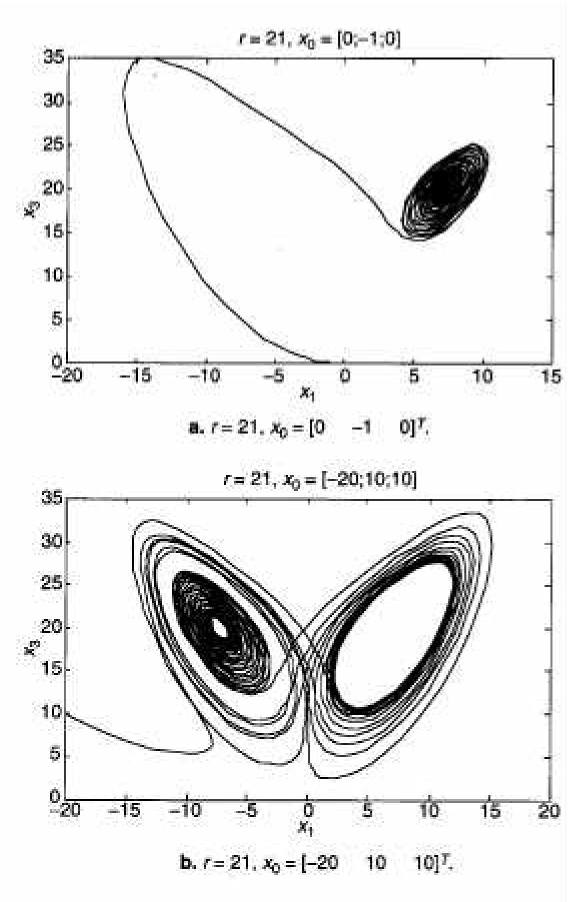
- Eigenvalues of nontrivial solutions: → stable

$$\lambda_1 = -13.4266; \lambda_2 = -0.12 + 8.9123i; \lambda_3 = -0.12 - 8.9123i$$

Transient response



Phase plane (x_1 vs x_3)



(3) For a chaotic conditions

(for σ=10, b=8/3, r=28)

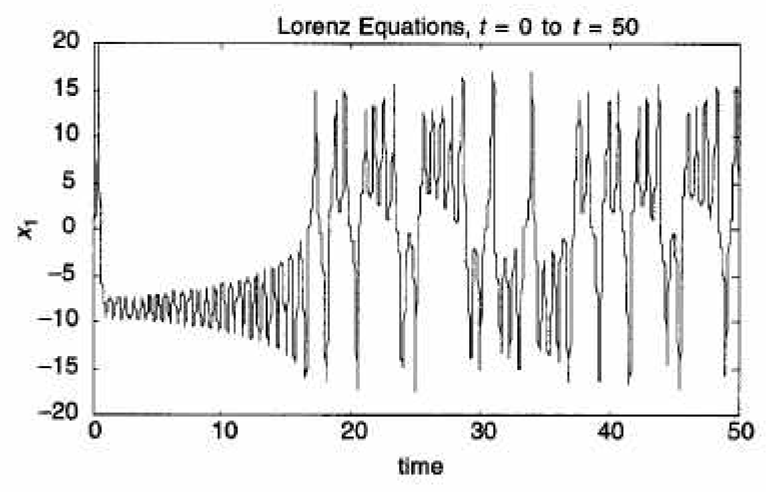
- *In this case, eigenvalues of trivial solution:* $\lambda_1 = -2.67; \lambda_2 = -22.83; \lambda_3 = 11.83$

→ *unstable*

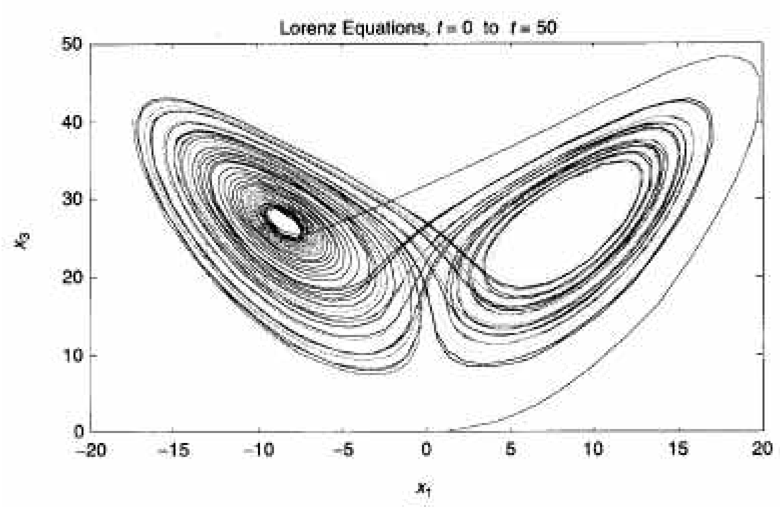
- Eigenvalues of nontrivial solutions: → unstable (strange attractor)

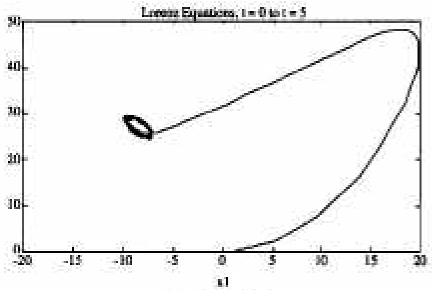
$$\lambda_1 = -13.8546; \lambda_2 = 0.094 + 10.1945i; \lambda_3 = 0.094 - 10.1945i$$

Transient response

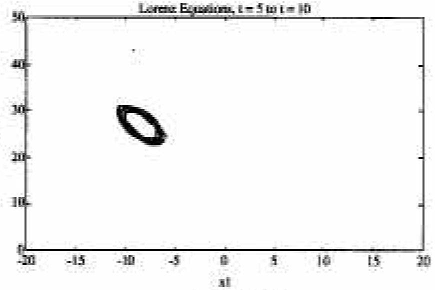


Phase plane (x_1 vs x_3)

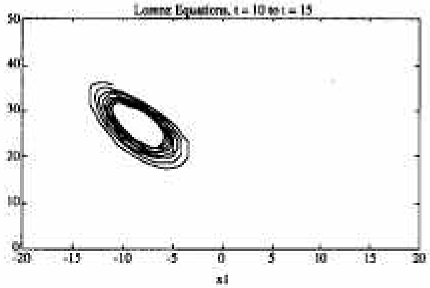




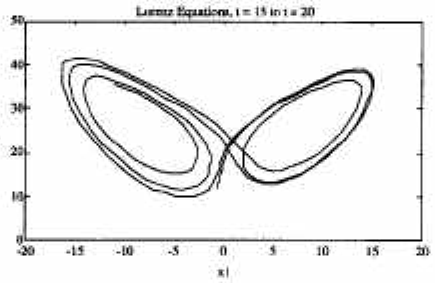
t = 0 to 5



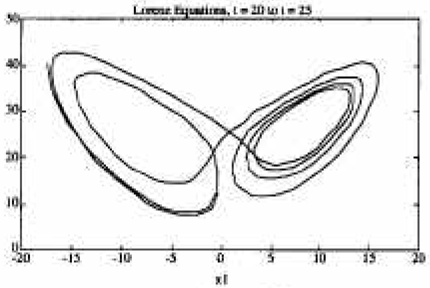
t = 5 to 10



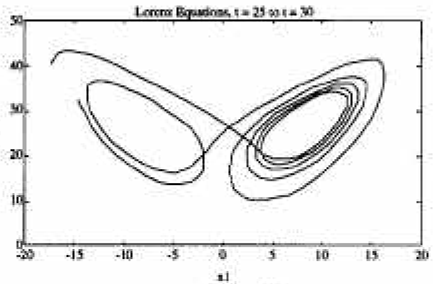
t = 10 to 15



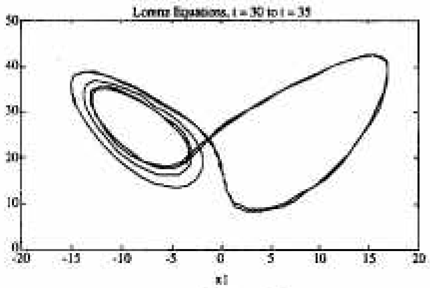
t = 15 to 20



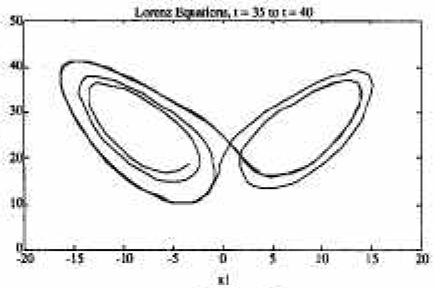
t = 20 to 25



t = 25 to 30



t = 30 to 35



t = 35 to 40