편미분방정식
(Partial Differential Equation, PDE)

Fourier Series

Now suppose that $f(x)$ is a given function of period 2π and is such that it can be **represented** by a series (4), that is, (4) converges and, moreover, has the sum $f(x)$. Then, using the equality sign, we write

(5)
$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

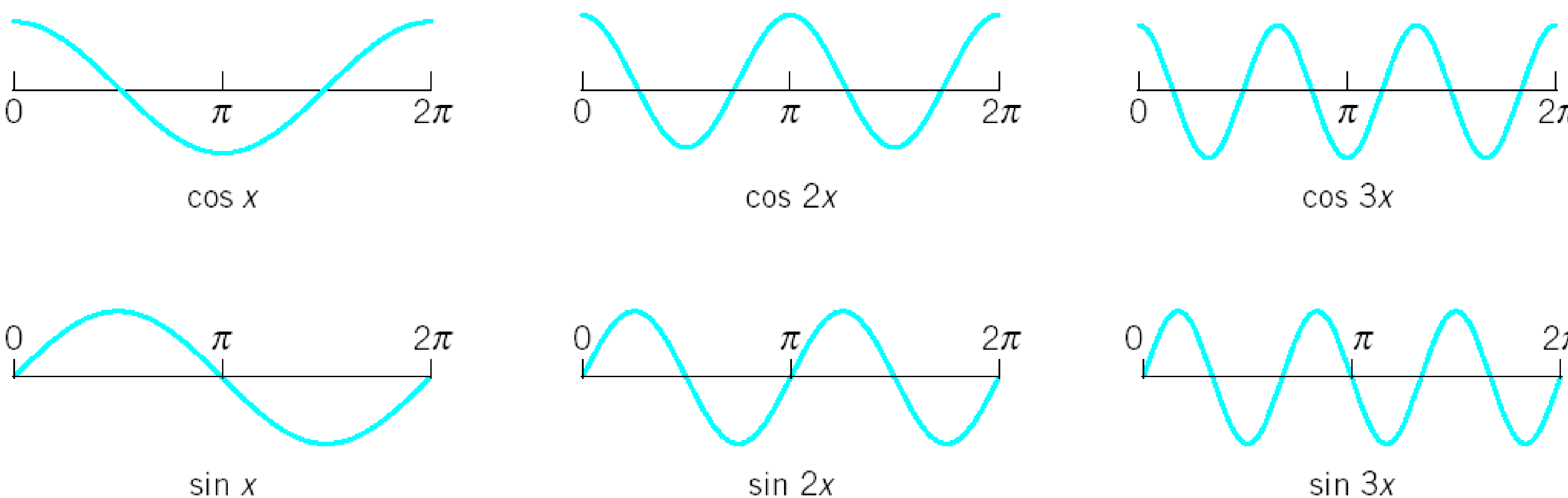


Fig. 256. Cosine and sine functions having the period 2π

and call (5) the **Fourier series** of $f(x)$. We shall prove that in this case the coefficients of (5) are the so-called **Fourier coefficients** of $f(x)$, given by the **Euler formulas**

(6)

(a) $$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

(b) $$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx \qquad n = 1, 2, \cdots$$

(c) $$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx \qquad n = 1, 2, \cdots .$$

we thus obtain from (1) the **Fourier series** of the function $f(x)$ of period $2L$

(5)
$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x \right)$$

with the **Fourier coefficients** of $f(x)$ given by the **Euler formulas**

(6)

(a)
$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\, dx$$

(b)
$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx \qquad n = 1, 2, \cdots$$

(c)
$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx \qquad n = 1, 2, \cdots .$$

Fourier Series Example
(Sawtooth wave)

Find the Fourier series of the function

$$f(x) = x + \pi \text{ if } \quad -\pi < x < \pi$$

and $f(x+2\pi) = f(x)$

Solution of Fourier series

Euler formula

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} x\sin nx dx = -\frac{2}{n}\cos n\pi$$

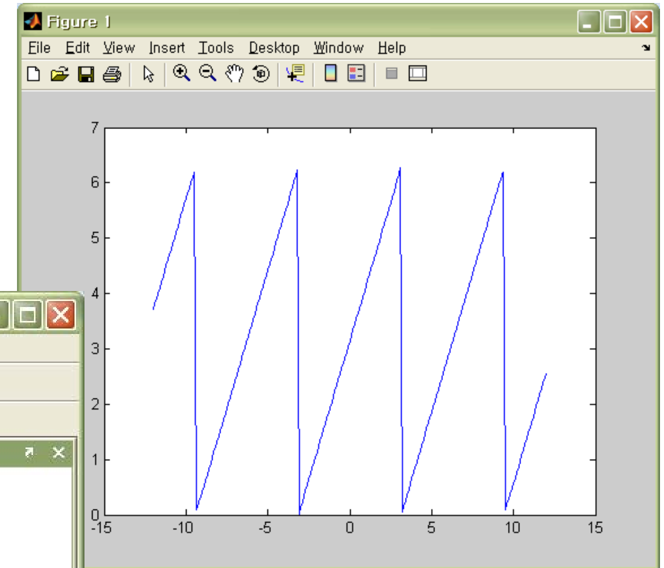
$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{L}x$$

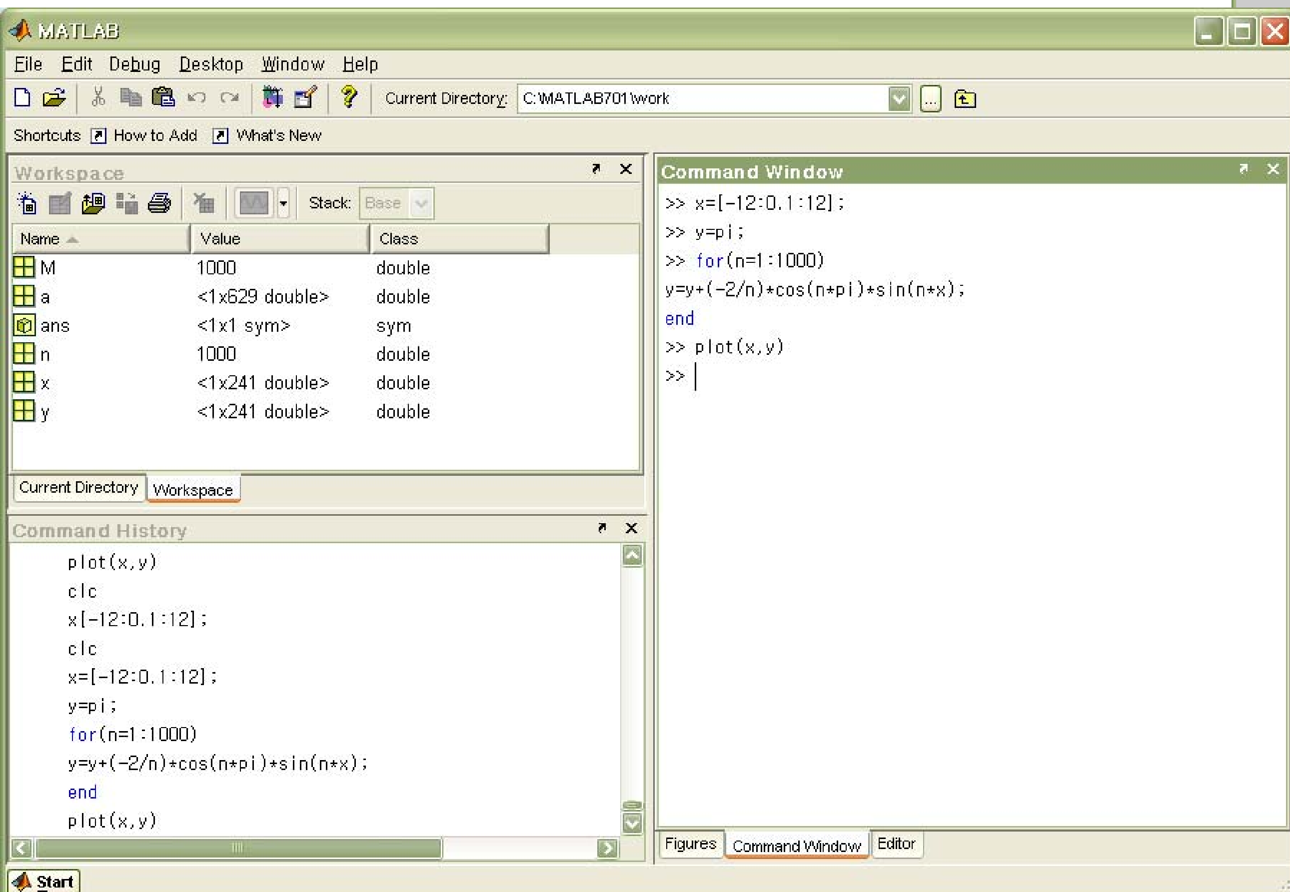
$$f(0) = \pi$$

$$f(x) = \pi + 2(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - + \cdots)$$

Graph of Function f(x) by Matlab

```
>> x=[-12:0.1:12];
>> y=pi;
>> for(n=1:1000)
y=y+(-2/n)*cos(n*pi)*sin(n*x);
end
>> plot(x,y)
```



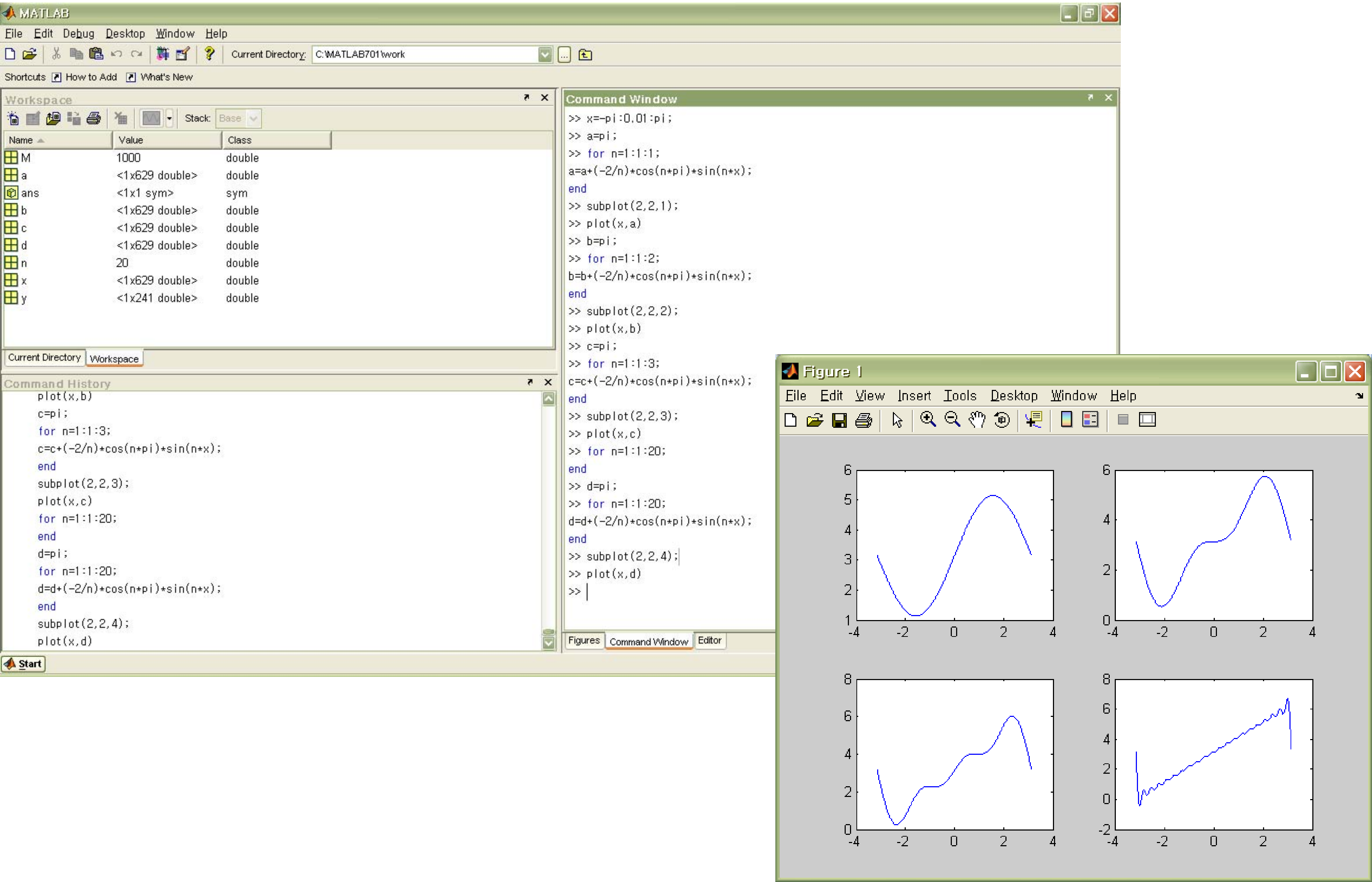


Matlab Program for Partial Sums S1,S2,S3,S20

```
>> x=-pi:0.01:pi;
>> a=pi;
>> for n=1:1:1;
a=a+(-2/n)*cos(n*pi)*sin(n*x);
end
>> subplot(2,2,1);
>> plot(x,a)
>> b=pi;
>> for n=1:1:2;
b=b+(-2/n)*cos(n*pi)*sin(n*x);
end
>> subplot(2,2,2);
>> plot(x,b)
```

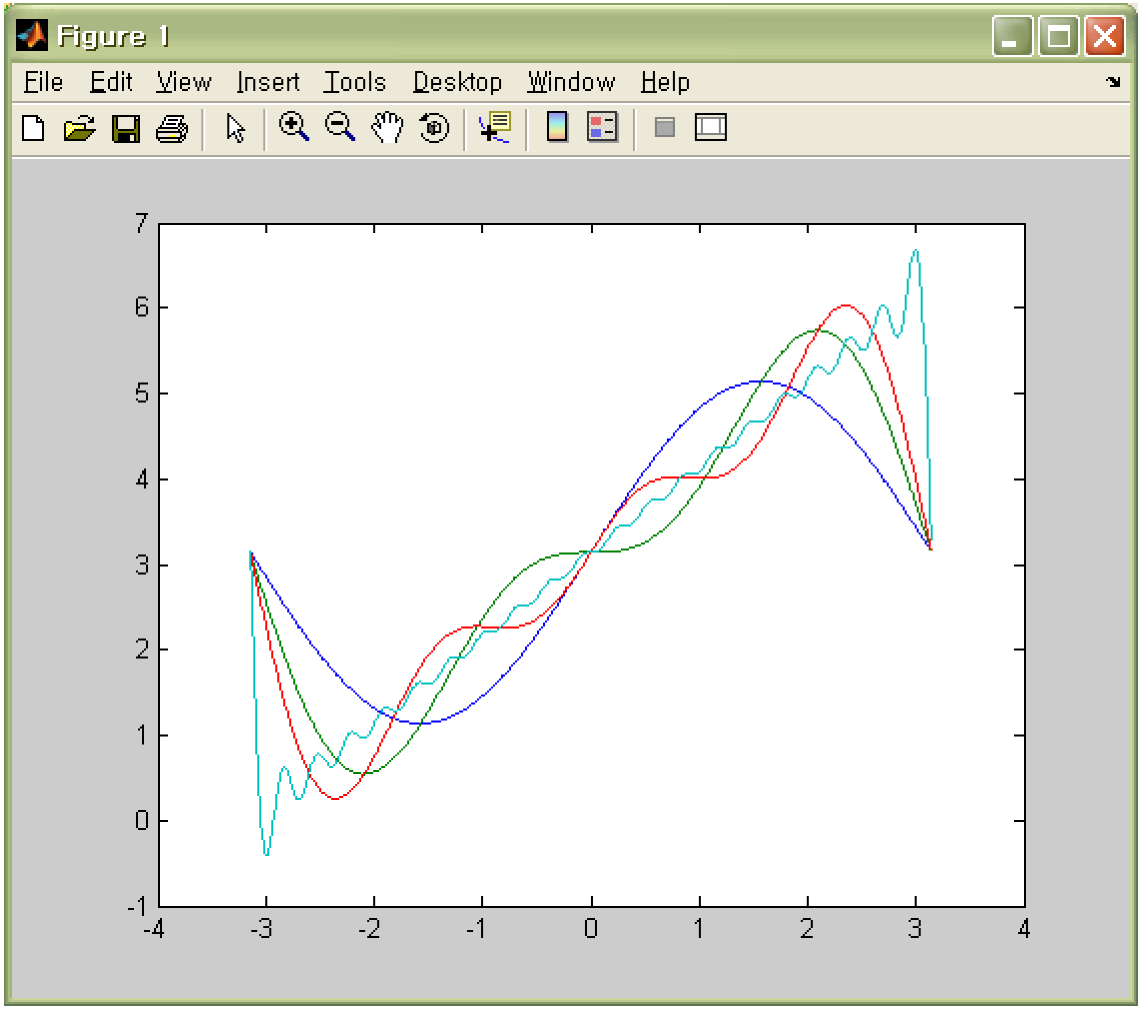
```
>> c=pi;
>> for n=1:1:3;
c=c+(-2/n)*cos(n*pi)*sin(n*x);
end
>> subplot(2,2,3);
>> plot(x,c)
>> d=pi;
>> for n=1:1:20;
d=d+(-2/n)*cos(n*pi)*sin(n*x);
end
>> subplot(2,2,4);
>> plot(x,d)
```

Partial sums S1,S2,S3,S20



Partial sums S1,S2,S3,S20

- >> x=-pi:0.01:pi;
- >> a=pi;
- >> for n=1:1:1;
- a=a+(-2/n)*cos(n*pi)*sin(n*x);
- end
- >> b=pi;
- >> for n=1:1:2;
- b=b+(-2/n)*cos(n*pi)*sin(n*x);
- end
- >> c=pi;
- >> for n=1:1:3;
- c=c+(-2/n)*cos(n*pi)*sin(n*x);
- end
- >> d=pi;
- >> for n=1:1:20;
- d=d+(-2/n)*cos(n*pi)*sin(n*x);
- end
- >> plot(x,[a; b; c; d])



A **partial differential equation (PDE)** is an equation involving one or more partial derivatives of an (unknown) function, call it u, that depends on two or more variables, often time t and one or several variables in space. The order of the highest derivative is called the **order** of the PDE. As for ODEs, second-order PDEs will be the most important ones in applications.

Just as for ordinary differential equations (ODEs) we say that a PDE is **linear** if it is of the first degree in the unknown function u and its partial derivatives. Otherwise we call it **nonlinear.** We call a *linear* PDE **homogeneous** if each of its terms contains either u or one of its partial derivatives. Otherwise we call the equation **nonhomogeneous.**

(1) $$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$ *One-dimensional wave equation*

(2) $$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$ *One-dimensional heat equation*

(3) $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$ *Two-dimensional Laplace equation*

The model of a vibrating elastic string (a violin string, for instance) consists of the **one-dimensional wave equation**

$$\text{(1)} \qquad \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad\qquad c^2 = \frac{T}{\rho}$$

for the unknown deflection $u(x, t)$ of the string, a PDE that we have just obtained, and some ***additional conditions,*** which we shall now derive.

Since the string is fastened at the ends $x = 0$ and $x = L$, we have the two **boundary conditions**

$$\text{(2)} \qquad \text{(a)} \quad u(0, t) = 0, \qquad \text{(b)} \quad u(L, t) = 0 \qquad \text{for all } t.$$

Furthermore, the form of the motion of the string will depend on its *initial deflection* (deflection at time $t = 0$), call it $f(x)$, and on its *initial velocity* (velocity at $t = 0$), call it $g(x)$. We thus have the two **initial conditions**

$$\text{(3)} \qquad \text{(a)} \quad u(x, 0) = f(x), \qquad \text{(b)} \quad u_t(x, 0) = g(x) \qquad (0 \leqq x \leqq L)$$

where $u_t = \partial u/\partial t$. We now have to find a solution of the PDE (1) satisfying the conditions (2) and (3). This will be the solution of our problem. We shall do this in three steps, as follows.

Separation of Variables and Fourier Series

Step 1. By the **"method of separating variables"** or *product method,* setting $u(x, t) = F(x)G(t)$, we obtain from (1) two ODEs, one for $F(x)$ and the other one for $G(t)$.

Step 2. We determine solutions of these ODEs that satisfy the boundary conditions (2).

Step 3. Finally, using **Fourier series,** we compose the solutions gained in Step 2 to obtain a solution of (1) satisfying both (2) and (3), that is, the solution of our model of the vibrating string.

PDE Wave Equation Analytical Solution

Wave Equation with the following initial condition for the string fixed at both ends at x = 0 and L

$$f(x) = \begin{cases} \dfrac{2k}{L}x & (0 < x < \dfrac{L}{2}) \\ \dfrac{2k}{L}(L-x) & (\dfrac{L}{2} < x < L) \end{cases}$$

12.3절 (예제1)

$f(x) \begin{cases} \frac{2k}{L}x & (0<x<\frac{L}{2}) \\ \frac{2k}{L}(L-x) & (\frac{L}{2}<x<L) \end{cases}$, 초기속도 $g(x)=0$

파동방정식 (1)의 해 구하라

(풀이) 파동방정식 (1) $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$, $c^2 = \frac{T}{\rho}$

* boundary condition
  $u(0,t)=0$, $u(L,t)=0$

* initial condition
  $u(x,0)=f(x)$ $u_t(x,0)=g(x)$

(1단계) 상미분방정식유도

$u(x,t)=F(x)G(t)$

파동방정식에의해

$\frac{\partial^2 u}{\partial t^2} = F\ddot{G}$ $\frac{\partial^2 u}{\partial x^2} = F''G$. $\Rightarrow$ $F\ddot{G} = c^2F''G$ (c^2FG로나누면)

$\frac{\ddot{G}}{c^2G} = \frac{F''}{F} = k$ (좌변은 t에만의존, 우변은 x에만의존)

$F''-kF=0$, $\ddot{G}-c^2kG=0$

(2단계) 경계조건의 만족

$u(0,t) = F(0)G(t) = 0$. $u(L,t) = F(L)G(t) = 0$.

$G \neq 0$.이다 $\Rightarrow F(0) = F(L) = 0$.

k가 음수일 때. $k = -p^2$

$F'' + p^2 F = 0$ 일반해 $F(x) = A\cos px + B\sin px$.

$F(0) = A = 0$. $F(L) = B\sin pL$ ($F \neq 0$ 이고 $B \neq 0$ 이므로 $p = \frac{n\pi}{L}$)

$k = -p^2 = -\left(\frac{n\pi}{L}\right)^2$

$F(x) = B\sin\frac{n\pi x}{L}$ (B=1일때) $F_n(x) = \sin\frac{n\pi}{L}x$. $(n=1,2,\cdots)$

$\ddot{G} + \lambda_n^2 G = 0$. $\left(\lambda_n = cp = \frac{cn\pi}{L}\right)$

$G_n(t) = B_n\cos\lambda_n t + B_n^* \sin\lambda_n t$

(eigenfunction)

$u_n(x,t) = F_n(x)\,G_n(t) = G_n(t) * F_n(x)$ → 고유함수

$= (B_n\cos\lambda_n t + B_n^* \sin\lambda_n t)\sin\frac{n\pi}{L}x$.

(3단계) 전체에 대한해 푸리에급수

$\lambda_n = \frac{cn a}{L}$

$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} (B_n \cos\lambda_n t + B_n^* \sin\lambda_n t) \sin\frac{n a x}{L}$

Initial condition 의 충족

② $u(x,0) = f(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n a x}{L}$

① $\left.\frac{\partial u}{\partial t}\right|_{t=0} = \left[\sum_{n=1}^{\infty} (-B_n\lambda_n \sin\lambda_n t + B_n^*\lambda_n \cos\lambda_n t) \sin\frac{n a x}{L} \right]_{t=0}$

$= \sum_{n=1}^{\infty} B_n^*\lambda_n \sin\frac{n a x}{L} = g(x)$

$g(x)=0$ 이므로 $B_n^* = 0$, $\lambda_n = \frac{cn a}{L}$

$\therefore u(x,t) = \sum_{n=1}^{\infty} B_n \cos\lambda_n t \cdot \sin\frac{n a x}{L}$ $\quad \lambda = \frac{cn a}{L}$

$= \sum_{n=1}^{\infty} \frac{1}{2} B_n \left(\sin(x+ct)\frac{n a}{L} + \sin(x-ct)\frac{n a}{L} \right)$

$= \frac{1}{2}\{ f^*(x+ct) + f^*(x-ct) \}$

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n a x}{L}dx$$ (무리에 사인급수되도록 Bn선정)

$$= \frac{2}{L}\left(\int_0^{\frac{L}{2}} \frac{2k}{L} x \sin\frac{n a x}{L}dx + \int_{\frac{L}{2}}^{L} \frac{2k}{L}(L-x)\sin\frac{n a x}{L}dx\right)$$

$$= \frac{4k}{L^2}\left(\underbrace{\int_0^{\frac{L}{2}} x\sin\frac{n a x}{L}dx}_{①} + \underbrace{\int_{\frac{L}{2}}^{L}(L-x)\sin\frac{n a x}{L}dx}_{②}\right)$$

① $$\int_0^{\frac{L}{2}} x\sin\frac{n a x}{L}dx = \left[x\left(-\frac{L}{na}\right)\cos\frac{na}{L}x\right]_0^{\frac{L}{2}} - \left(-\frac{L}{na}\right)\int_0^{\frac{L}{2}}\cos\frac{n a x}{L}dx$$

$$= \left(-\frac{L^2}{2na}\cos\frac{na}{2} - 0\right) + \left(\frac{L}{na}\right)\left(\frac{L}{na}\right)\sin\frac{n a x}{L}\Big|_0^{\frac{L}{2}}$$

$$= -\frac{L^2}{2na}\cos\frac{na}{2} + \left(\frac{L}{na}\right)^2\left(\sin\frac{na}{2} - 0\right)$$

② $$\int_{\frac{L}{2}}^{L}(L-x)\sin\frac{n a x}{L}dx = \left[(L-x)\left(-\frac{L}{na}\right)\cos\frac{n a x}{L}\right]_{\frac{L}{2}}^{L} - \left(-\frac{L}{na}\right)\int_{\frac{L}{2}}^{L}\cos\frac{na}{L}(-1)\,dx$$

$$= \left(0 - \left(\frac{L}{2}\right)\left(-\frac{L}{na}\right)\cos\frac{na}{2}\right) - \left(\frac{L}{na}\right)\left(\frac{L}{na}\right)\sin\frac{na}{L}\Big|_{\frac{L}{2}}^{L}$$

$$= \frac{L^2}{2na}\cos\frac{na}{2} - \left(\frac{L}{na}\right)^2\left(\underbrace{\sin na}_{\to 0} - \sin\frac{na}{2}\right)$$

$$①+② = 2\left(\frac{L}{na}\right)^2\sin\frac{na}{2}$$

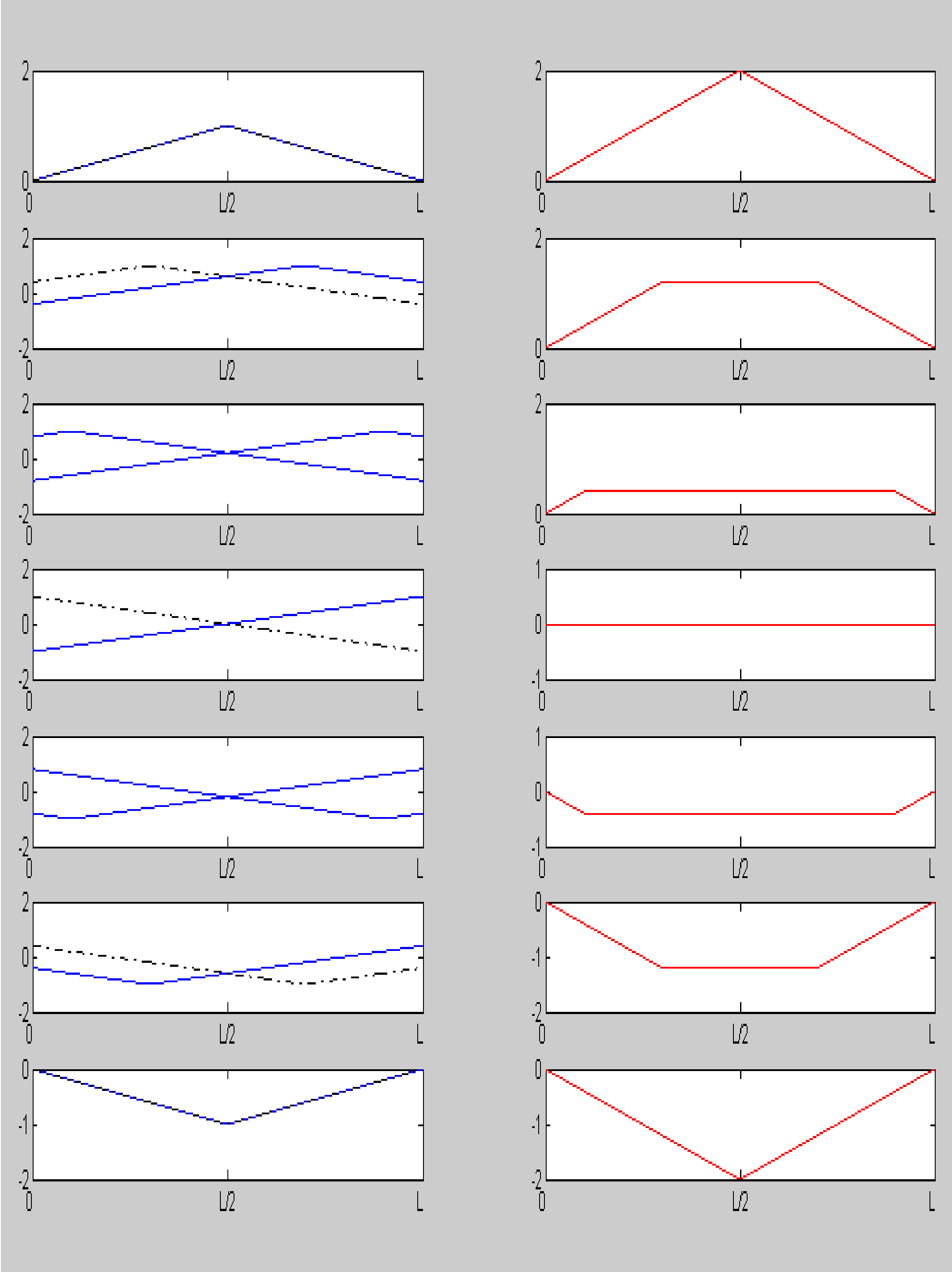
Matlab Program for Wave Equation Solution

Command Window

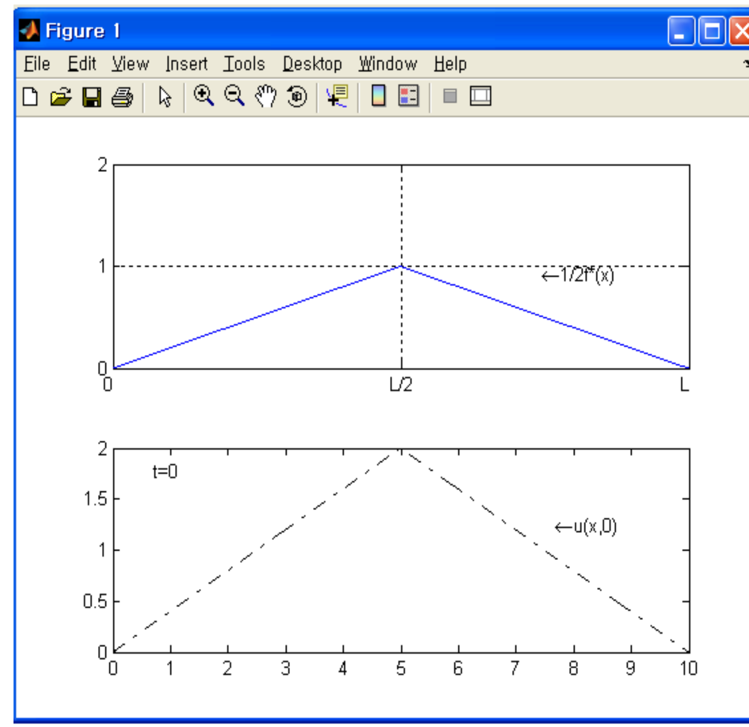
```
>> L=10;
>> x = 0 : 1/1000 :L;
>> c=1;
>> t=0;
>> k=2;
>> u1=0;
>> for n = 1: 1 : 1000
g = 8*k/((n*pi)^2)*sin(n*pi/2);
h = sin(n*pi/L*(x-c*t));
u1 = u1 +1/2*g*h;
end
subplot(2,1,1), plot(x,u1,'r');
>> hold on;
>> u2 = 0;
>> for n = 1:1:1000
g = 8*k/((n*pi)^2)*sin(n*pi/2);
h = sin(n*pi/L*(x+c*t));
u2 = u2 + 1/2*g*h;
end
>> subplot(2,1,1),plot(x,u2,'b');
>> axis([0 L 0 2]);
>> set(gca,'xtick',[0:L/2:L]);
x_label=str2mat('0','L/2','L');
>> set(gca,'xticklabel',x_label);
>> set(gca,'ytick',[0:1:2]);
>> gtext('₩leftarrow1/2f*(x)');
>> grid on;
>> set(gcf,'color','₩');
>> u=0;
>> u= u1+u2;
>> subplot(2,1,2),plot(x,u,'-.k');
>> gtext('₩leftarrowu(x,0)');
>> gtext('t=0');
>>
```

$$u(x,t) = \frac{1}{2} f^*(x-ct) + \frac{1}{2} f^*(x+ct)$$

$$\sum u_1 = \frac{1}{2} f^*(x-ct) \quad \& \quad \sum u_2 = \frac{1}{2} f^*(x+ct)$$



Graphs from the Matlab Program



Heat or Diffusion Equation

From the wave equation we now turn to the next "big" PDE, the **heat equation**

$$\frac{\partial u}{\partial t} = c^2 \nabla^2 u, \qquad c^2 = \frac{K}{\sigma\rho},$$

which gives the temperature $u(x, y, z, t)$ in a body of homogeneous material. Here c^2 is the thermal diffusivity, K the thermal conductivity, σ the specific heat, and ρ the density of the material of the body. $\nabla^2 u$ is the Laplacian of u, and with respect to Cartesian coordinates x, y, z,

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}.$$

The heat equation was derived in Sec. 10.8. It is also called the **diffusion equation.**

As an important application, let us first consider the temperature in a long thin metal bar or wire of constant cross section and homogeneous material, which is oriented along the x-axis (Fig. 291) and is perfectly insulated laterally, so that heat flows in the x-direction

Furthermore, the initial temperature in the bar at time $t = 0$ is given, say, $f(x)$, so that we have the **initial condition**

(3) $$u(x, 0) = f(x) \qquad [f(x) \text{ given}].$$

Here we must have $f(0) = 0$ and $f(L) = 0$ because of (2).

We shall determine a solution $u(x, t)$ of (1) satisfying (2) and (3)—one initial condition will be enough, as opposed to two initial conditions for the wave equation. Technically, our method will parallel that for the wave equation, a separation of variables, followed by the use of Fourier series. You may find a step-by-step comparison worthwhile.

***Step 1.* Two ODEs from the heat equation (1).** Substitution of a product $u(x, t) = F(x)G(t)$ into (1) gives $F\dot{G} = c^2F''G$ with $\dot{G} = dG/dt$ and $F'' = d^2F/dx^2$ To separate the variables, we divide by c^2FG, obtaining

(4) $$\frac{\dot{G}}{c^2G} = \frac{F''}{F}$$

The left side depends only on t and the right side only on x, so that both sides must equal a constant k. You may show that for $k = 0$ or $k > 0$ the only solution $u = FG$ satisfying (2) is $u \equiv 0$. For negative $k = -p^2$ we have from (4)

$$\frac{\dot{G}}{c^2G} = \frac{F''}{F} = -p^2$$

Multiplication by the denominators gives immediately the two ODEs

(5) $$F'' + p^2F = 0$$

and

$$\dot{G} + c^2p^2G = 0. \tag{6}$$

***Step 2.* Satisfying the boundary conditions (2).** We first solve (5). A general solution is

$$F(x) = A\cos px + B\sin px. \tag{7}$$

From the boundary conditions (2) it follows that

$$u(0, t) = F(0)G(t) = 0 \qquad \text{and} \qquad u(L, t) = F(L)G(t) = 0.$$

Since $G \equiv 0$ would give $u \equiv 0$, we require $F(0) = 0$, $F(L) = 0$ and get $F(0) = A = 0$ by (7) and then $F(L) = B\sin pL = 0$, with $B \neq 0$ (to avoid $F \equiv 0$); thus,

$$\sin pL = 0, \qquad \text{hence} \qquad p = \frac{n\pi}{L}, \qquad n = 1, 2, \cdots$$

Setting $B = 1$, we thus obtain the following solutions of (5) satisfying (2):

$$F_n(x) = \sin\frac{n\pi x}{L}, \qquad n = 1, 2, \cdots$$

$$\dot{G} + \lambda_n{}^2 G = 0 \qquad \text{where} \qquad \lambda_n = \frac{cn\pi}{L}$$

It has the general solution

$$G_n(t) = B_n e^{-\lambda_n{}^2 t}, \qquad n = 1, 2, \cdots$$

where B_n is a constant. Hence the functions

$$u_n(x, t) = F_n(x)G_n(t) = B_n \sin \frac{n\pi x}{L} e^{-\lambda_n{}^2 t} \qquad (n = 1, 2, \cdots) \tag{8}$$

are solutions of the heat equation (1), satisfying (2). These are the **eigenfunctions** of the problem, corresponding to the **eigenvalues** $\lambda_n = cn\pi/L$.

Step 3. **Solution of the entire problem. Fourier series.** So far we have solutions (8) satisfying the boundary conditions (2). To obtain a solution that also satisfies the initial condition (3), we consider a series of these eigenfunctions,

$$u(x, t) = \sum_{n=1}^{\infty} u_n(x, t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} e^{-\lambda_n{}^2 t} \qquad \left(\lambda_n = \frac{cn\pi}{L}\right). \tag{9}$$

From this and (3) we have

$$u(x, 0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} = f(x).$$

Hence for (9) to satisfy (3), the B_n's must be the coefficients of the **Fourier sine series,** as given by (4) in Sec. 11.3; thus

(10)
$$B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx \qquad (n = 1, 2, \cdots).$$

Bar with Insulated Ends. Eigenvalue 0

Find a solution formula of (1), (3) with (2) replaced by the condition that both ends of the bar are insulated.

Solution. Physical experiments show that the rate of heat flow is proportional to the gradient of the temperature. Hence if the ends $x = 0$ and $x = L$ of the bar are insulated, so that no heat can flow through the ends, we have grad $u = u_x = \partial u/\partial x$ and the boundary conditions

$$u_x(0, t) = 0, \qquad u_x(L, t) = 0 \qquad \text{for all } t. \tag{2*}$$

Since $u(x, t) = F(x)G(t)$, this gives $u_x(0, t) = F'(0)G(t) = 0$ and $u_x(L, t) = F'(L)G(t) = 0$. Differentiating (7), we have $F'(x) = -Ap \sin px + Bp \cos px$, so that

$$F'(0) = Bp = 0 \qquad \text{and then} \qquad F'(L) = -Ap \sin pL = 0.$$

The second of these conditions gives $p = p_n = n\pi/L$, $(n = 0, 1, 2, \cdots)$. From this and (7) with $A = 1$ and $B = 0$ we get $F_n(x) = \cos(n\pi x/L)$, $(n = 0, 1, 2, \cdots)$. With G_n as before, this yields the eigenfunctions

$$u_n(x, t) = F_n(x) G_n(t) = A_n \cos \frac{n\pi x}{L}\, e^{-\lambda_n^2 t} \qquad (n = 0, 1, \cdots) \tag{11}$$

corresponding to the eigenvalues $\lambda_n = cn\pi/L$. The latter are as before, but we now have the additional eigenvalue $\lambda_0 = 0$ and eigenfunction $u_0 = const$, which is the solution of the problem if the initial temperature $f(x)$ is constant. This shows the remarkable fact that ***a separation constant can very well be zero, and zero can be an eigenvalue.***

Furthermore, whereas (8) gave a Fourier sine series, we now get from (11) a Fourier cosine series

(12) $$u(x, t) = \sum_{n=0}^{\infty} u_n(x, t) = \sum_{n=0}^{\infty} A_n \cos \frac{n\pi x}{L} \, e^{-\lambda_n{}^2 t} \qquad \left(\lambda_n = \frac{cn\pi}{L}\right).$$

Its coefficients result from the initial condition (3),

$$u(x, 0) = \sum_{n=0}^{\infty} A_n \cos \frac{n\pi x}{L} = f(x),$$

in the form (2), Sec. 11.3, that is,

(13) $$A_0 = \frac{1}{L} \int_0^L f(x)\, dx, \qquad A_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L}\, dx, \qquad n = 1, 2, \cdots .$$