10. Converging Flow

10.1 Introduction

Consider

* the converging or diverging flow
* exact solution exists for flow in a cone
  or for flow between flat plates

The planar converging flow or Hamel flow :

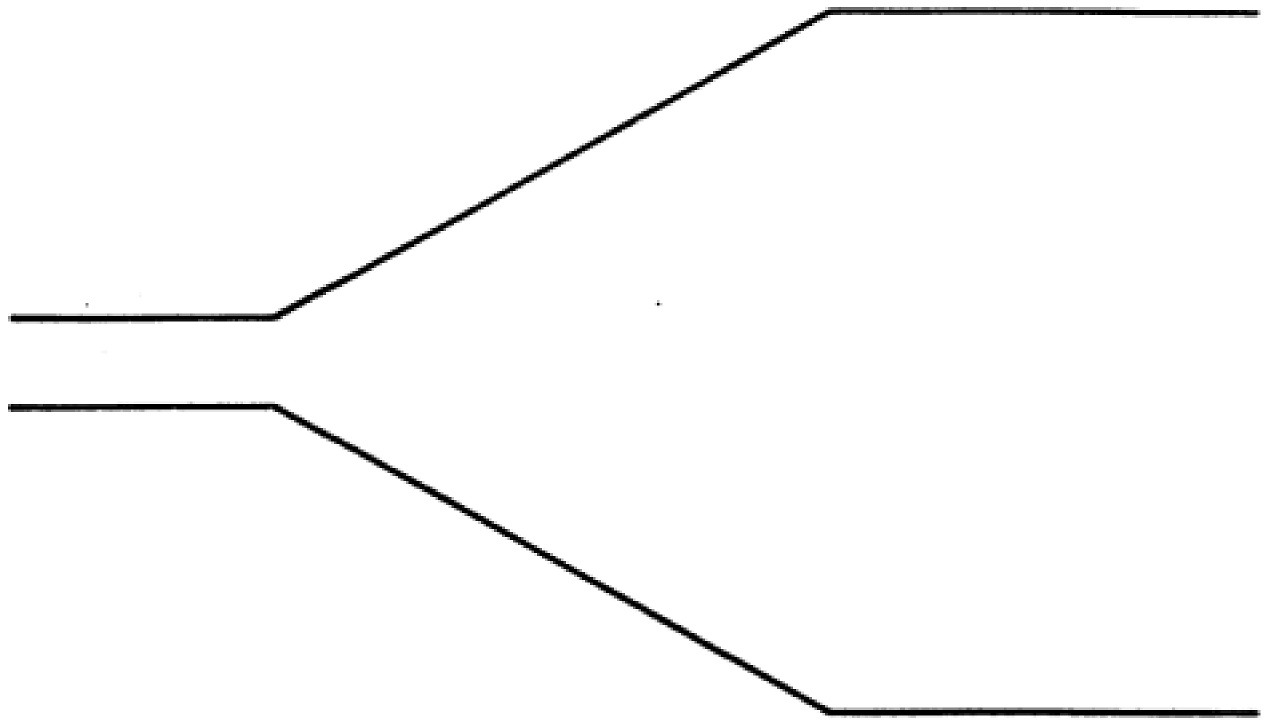

Fig. 10-1. Schematic of a finite converging or diverging flow.

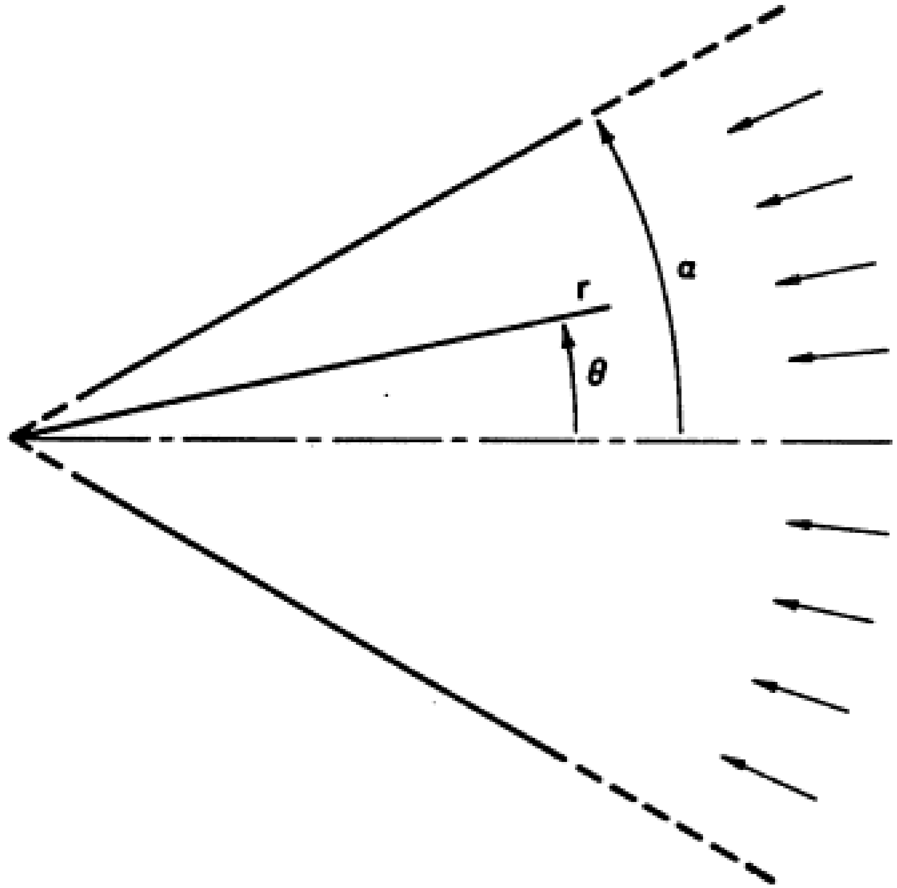

Fig. 10-2. Schematic of an infinite converging or diverging flow.

10.2 Solution

Kinematic assumptions:

* the flow is entirely radial, $\upsilon_z = \upsilon_\theta = 0$
* no variations in z direction $\Rightarrow$ $\partial / \partial z = 0$

$$\therefore \quad \upsilon_r = \upsilon_r(r,\theta)$$

The continuity:

* $$\frac{1}{r}\frac{\partial}{\partial r}(r\upsilon_r) = 0$$

$\Rightarrow$ $r\upsilon_r$ = function only of θ = $f(\theta)$ or $\upsilon_r = \dfrac{f(\theta)}{r}$

The boundary and flow conditions:

* $f(+\alpha) = f(-\alpha) = 0$ $\Leftarrow$ no-slip at the side walls
* the flow rate per unit width, q

$$q = \int_{-\alpha}^{+\alpha} \upsilon_r r d\theta = \int_{-\alpha}^{+\alpha} f(\theta) d\theta$$

Navier-Stokes eq'ns:

* r component

$$\rho \upsilon_r \frac{\partial \upsilon_r}{\partial r} = -\frac{\partial P}{\partial r} + \eta [\frac{\partial}{\partial r} \frac{1}{r} \frac{\partial}{\partial r}(r\upsilon_r) + \frac{1}{r^2} \frac{\partial^2 \upsilon_r}{\partial \theta^2}]$$

* θ component $0 = -\frac{1}{r} \frac{\partial P}{\partial \theta} + \eta \frac{2}{r^2} \frac{\partial \upsilon_r}{\partial \theta}$

Substituting $f(\theta)/r$ for υ_r gives

$$-\rho \frac{f^2}{r^3} = -\frac{\partial P}{\partial r} + \frac{\eta}{r^3} \frac{d^2 f}{d\theta^2}$$

$$0 = -\frac{\partial P}{\partial \theta} + \frac{2\eta}{r^2} \frac{df}{d\theta}$$

Eliminating the pressure (P) by cross-differentiation and subtraction

$$\frac{-2\rho}{r^3} f \frac{df}{d\theta} = \frac{\eta}{r^3} \frac{d^3 f}{d\theta^3} + \frac{4\eta}{r^3} \frac{df}{d\theta}$$

or $$\left(\frac{2\rho}{\eta}\right) f \frac{df}{d\theta} + \frac{d^2 f}{d\theta^3} + 4\frac{df}{d\theta} = 0$$

: a third-order O.D.E., requiring three conditions

Scaling the variables, we define

$\phi = \frac{\theta}{\alpha}$: a normalized angle

$F = \frac{f}{q}$: a normalized flow variable

The governing eq'n and boundary conditions then become

$$RF\frac{dF}{d\phi} + \frac{d^3F}{d\phi^3} + 4\alpha^2\frac{dF}{d\phi} = 0$$

$$F(-1) = F(+1) = 0 \quad , \quad \int_{-1}^{+1} F(\phi)d\phi = 1$$

where $R = \frac{2\rho q\alpha}{\eta}$: the ratio of inertial to viscous stresses

* the range of R : $-\infty < R < \infty$
* R goes to zero both as $\rho q/\eta \to 0$ and as $\alpha \to 0$.

Analytic solution exists in terms of integrals that is evaluated numerically

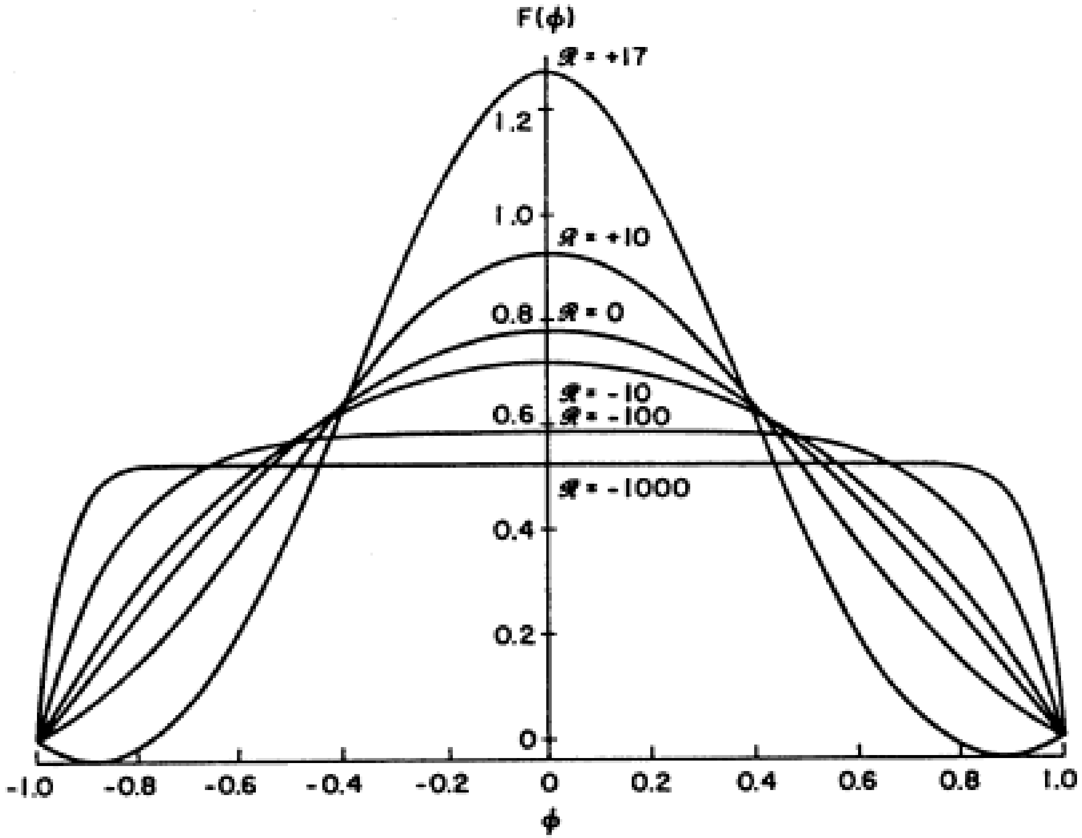

Fig. 10-3. Dimensionless velocity function $F(\phi)$, $\alpha = \pi/4$.

The numerical solution (Fig. 10-3 for $\alpha = \pi/4$) :

* The curves for $R = +1$ and $R = -1$ are indistinguishable from the curve for $R = 0$
* Very different behavior for inflow and outflow
* When R is large and negative (converging flow), υ_r is nearly constant over most of the included angle, approaching a value of $F(\phi) \approx 1/2$, and all the velocity variation is in a small boundary layer near the wall.
* When R is positive (diverging flow), $F(\phi)$ becomes negative near the wall for $R > 14$, indicating a backflow toward the vertex in this region.

  : flow separation

  : unstable turbulence occurs in practice

10.3 Orders of Magnitude

$$RF\frac{dF}{d\Phi}+\frac{d^3F}{d\Phi^3}+4\alpha^2\frac{dF}{d\Phi}=0$$

* F and derivatives : $O(1)$
* $R \to 0$: creeping flow approximation (Ch. 12)
* $\alpha \to 0$: lubrication approximation (Ch. 13)
* $|R| \to \infty$: boundary layer approximation (Ch. 15)

10.4 Creeping Flow, $R \to 0$

$$R \to 0 : \quad \frac{d^3F}{d\phi^3} + 4\alpha^2 \frac{dF}{d\phi} = 0$$

: a linear O.D.E. with constant coefficients

The general solution is

$$F(\phi) = A + B\sin 2\alpha\phi + C\cos 2\alpha\phi$$

After applying the boundary conditions,

$$F(\phi) = \frac{\alpha(\cos 2\alpha\phi - \cos 2\alpha)}{\sin 2\alpha - 2\alpha\cos 2\alpha} \qquad \text{for } \alpha \neq 0$$

$$F(\phi) = \frac{3}{4}(1-\phi^2) \qquad \text{for } \alpha = 0$$

The flows for which the inertial terms can be neglected are called creeping flows. ⇒ linearization

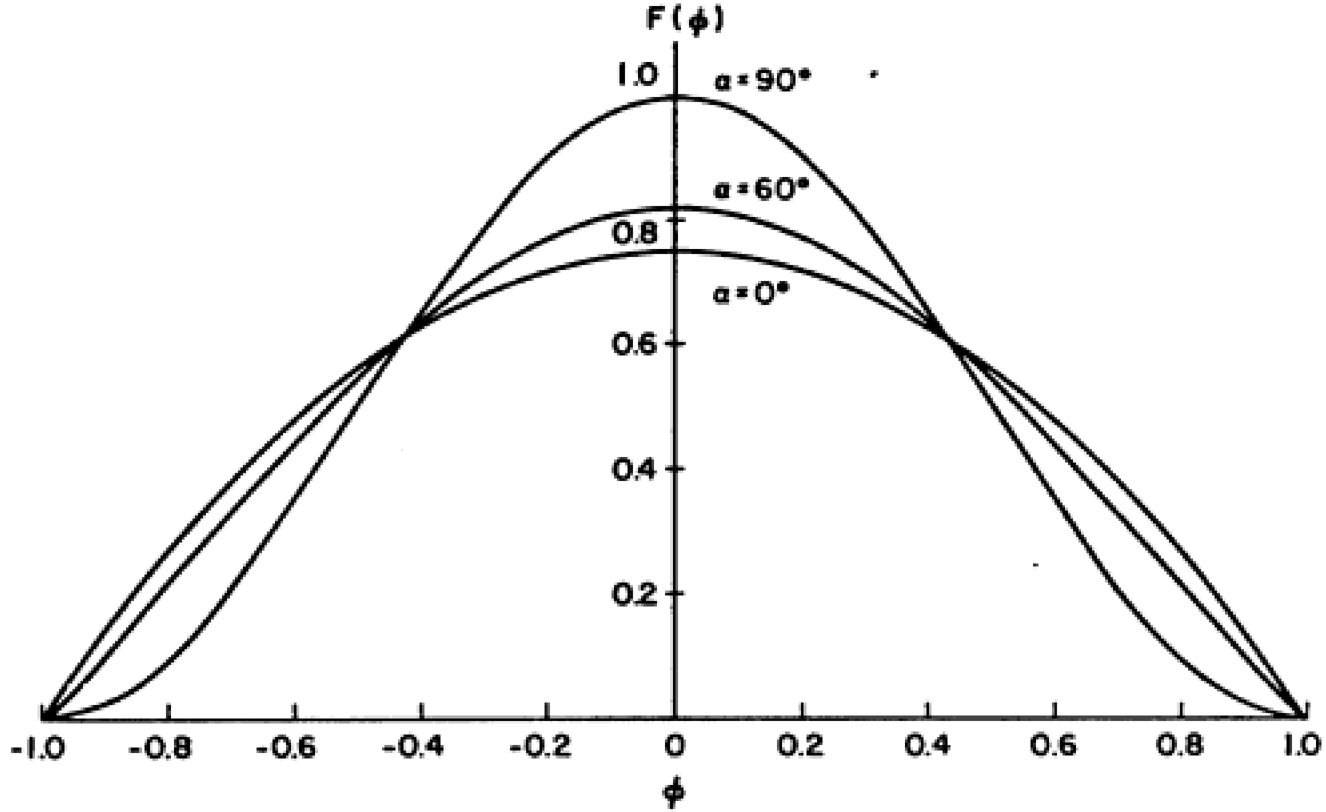

Fig. 10-4. Dimensionless velocity function $F(\phi)$ for $R = 0$.

10.5 Lubrication Approximation, $\alpha \to 0$

$\alpha \to 0$: $$\frac{d^3F}{d\phi^3} = 0$$

This has a solution of $F(\phi) = \frac{3}{4}(1-\phi^2)$

: a special case of the creeping flow approximation

The velocity profile:

$$v_r = \frac{f(\theta)}{r} = \frac{3q}{4(r\alpha)}[1-(\frac{r\theta}{r\alpha})^2]$$

or $$v_r \approx \frac{3q}{4(H/2)}[1-\frac{y^2}{(H/2)^2}]$$

where $y \approx r\theta$: distance from the centerline

$H(r) = 2r\sin\alpha \approx 2r\alpha$: the channel width at r

Since $q = \langle v_r \rangle H$,

$$v_r \approx \frac{3}{2} \langle v_r \rangle \left[1 - \left(\frac{2y}{H}\right)^2\right]$$

: identical to the flow between parallel plates

The lubrication approximation is therefore equivalent to treating the flow locally as though the flow were between parallel plates, but using the plate spacing H which is valid at that particular position.

10.6 Boundary Layer Approximation, $|R| \to \infty$

$|R| \to \infty$: $F\dfrac{dF}{d\Phi} = 0$

$\Rightarrow F =$ constant $= \dfrac{1}{2}$

: the flow for large negative R for the region away from the wall.

: cannot satisfy the no-slip condition at the wall.

: singular, the order of ODE drops from 3 to 1.

: describes the inviscid flow in the core.

There is a region (boundary layer) near the wall, penetrating into the fluid a distance proportional to $\sqrt{\eta/\rho}$, where the viscous terms must be considered.

The scaling variable for θ in the boundary layer:

$$\upsilon_r = \frac{dr}{dt} = \frac{f(\theta)}{r} = \frac{Fq}{\alpha r} \approx \frac{q}{2\alpha r} \qquad \Rightarrow \qquad t \approx \frac{\alpha r^2}{q}$$

$\therefore$ boundary layer thickness $= \delta \approx \sqrt{\dfrac{\eta t}{\rho}} = r\sqrt{\dfrac{\alpha\eta}{\rho|q|}}$

boundary layer angle $= \dfrac{\delta}{r} = \sqrt{\dfrac{\alpha\eta}{\rho|q|}} \approx \alpha|R|^{-1/2}$

Define a scaled angle:

$$\zeta = \frac{\theta+\alpha}{\alpha|R|^{-1/2}} = (\phi+1)|R|^{1/2} \qquad \Rightarrow \quad d\zeta = |R|^{1/2}d\phi$$

Then, $$RF\frac{dF}{d\phi} + \frac{d^3F}{d\phi^3} + 4\alpha^2\frac{dF}{d\phi} = 0$$

$$\Rightarrow \quad R|R|^{1/2}F\frac{dF}{d\zeta} + |R|^{3/2}\frac{d^3F}{d\zeta^3} + 4\alpha^2|R|^{1/2}\frac{dF}{d\phi} = 0$$

or $$\frac{R}{|R|}F\frac{dF}{d\zeta} + \frac{d^3F}{d\zeta^3} + \frac{4\alpha^2}{|R|}\frac{dF}{d\zeta} = 0$$

Let $|R| \to \infty$, then

$$R > 0: \qquad F\frac{dF}{d\zeta} + \frac{d^3F}{d\zeta^3} = 0$$

$$R < 0: \quad -F\frac{dF}{d\zeta} + \frac{d^3F}{d\zeta^3} = 0$$

: no longer singular

Boundary conditions:

$$\zeta = 0: \quad F = 0$$

$$\zeta \to \infty: \quad F \to \frac{1}{2} \qquad \frac{dF}{d\zeta} \to 0$$

$$R > 0 : \ F\frac{dF}{d\zeta} + \frac{d^3F}{d\zeta^3} = 0 \qquad \Rightarrow \qquad \frac{1}{2}\frac{dF^2}{d\zeta} + \frac{d^3F}{d\zeta^3} = 0$$

Integrating once, then

$$\frac{1}{2}F^2 + \frac{d^2F}{d\zeta^2} = \text{constant} = \frac{1}{2}F^2(\infty) = \frac{1}{8}$$

or $\quad \frac{1}{2}F^2\frac{dF}{d\zeta} + \frac{dF}{d\zeta}\frac{d^2F}{d\zeta^2} - \frac{1}{8}\frac{dF}{d\zeta} = 0 \quad ,$

$$\frac{1}{6}\frac{dF^3}{d\zeta} + \frac{1}{2}\frac{d}{d\zeta}\left(\frac{dF}{d\zeta}\right)^2 - \frac{1}{8}\frac{dF}{d\zeta} = 0 \quad ,$$

Integrating again, $\quad \frac{1}{6}F^3 + \frac{1}{2}\left(\frac{dF}{d\zeta}\right)^2 - \frac{1}{8}F = \text{constant}$

$$= \frac{1}{6}F^3(\infty) - \frac{1}{8}F(\infty) = -\frac{1}{24}$$

At $\zeta = 0$, we have $\left(\frac{dF}{d\zeta}\right)^2 = -\frac{1}{12}$ $\quad \therefore$ The solution of the type that we are seeking cannot exist for a diverging flow.

$$R < \mathrm{C} : \; -F\frac{dF}{d\zeta} + \frac{d^3F}{d\zeta^3} = 0 \qquad \Rightarrow \qquad -\frac{1}{2}\frac{dF^2}{d\zeta} + \frac{d^3F}{d\zeta^3} = 0$$

Integrating once, then

$$-\frac{1}{2}F^2 + \frac{d^2F}{d\zeta^2} = \text{constant} = -\frac{1}{2}F^2(\infty) = -\frac{1}{8}$$

or $$-\frac{1}{2}F^2\frac{dF}{d\zeta} + \frac{dF}{d\zeta}\frac{d^2F}{d\zeta^2} + \frac{1}{8}\frac{dF}{d\zeta} = 0 \quad ,$$

$$-\frac{1}{6}\frac{dF^3}{d\zeta} + \frac{1}{2}\frac{d}{d\zeta}\left(\frac{dF}{d\zeta}\right)^2 + \frac{1}{8}\frac{dF}{d\zeta} = 0 \quad ,$$

Integrating again,

$$-\frac{1}{6}F^3 + \frac{1}{2}\left(\frac{dF}{d\zeta}\right)^2 + \frac{1}{8}F = \text{constant}$$

$$= -\frac{1}{6}F^3(\infty) + \frac{1}{8}F(\infty) = +\frac{1}{24}$$

At $\zeta = 0$, we have $\left(\frac{dF}{d\zeta}\right)^2 = +\frac{1}{12}$: no problem

The solution is

$$F(\phi) = \frac{3}{2}\left[\frac{2.225e^{(\phi+1)\sqrt{|R|/2}} - 0.225}{2.225e^{(\phi+1)\sqrt{|R|/2}} + 0.225}\right]^2 - 1$$

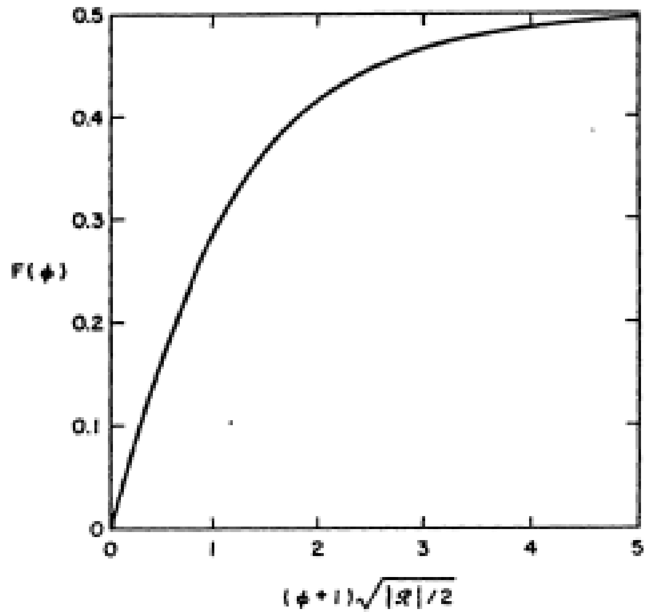

Fig. 10-5. Dimensionless velocity function $F(\phi)$ for $R \to -\infty$.

: $F \approx \frac{1}{2}$ at $(\phi+1)\sqrt{|R|/2} \to 5$ or $(\phi+1) \to 7/\sqrt{|R|}$

dimensionless angular boundary thickness

10.7 Power Requirement

* For flow in a straight pipe : power = flow rate × pressure drop

In Hamel flow :

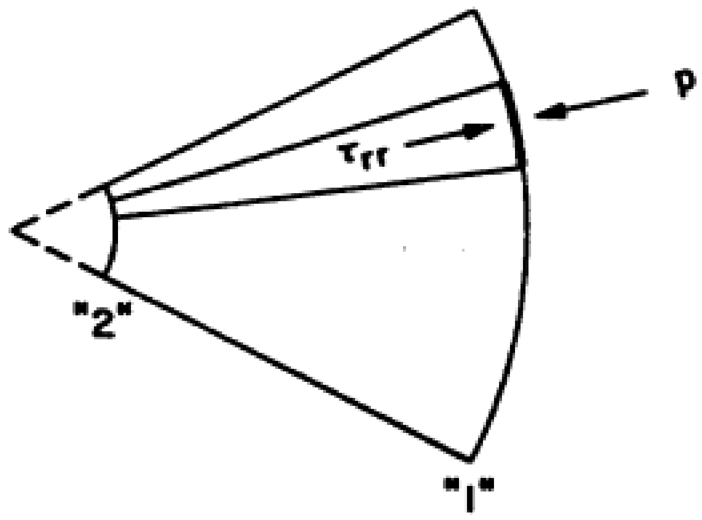

Fig. 10-6. Schematic of stresses and surfaces used in power calculation.

At surface 1, the force per unit distance into the screen on the arc sector $d\theta$ is $(p-\tau_{rr})rd\theta$. Thus, the power to move fluid across that sector of the surface is $\upsilon_r(p-\tau_{rr})rd\theta$. Therefore, the total power requirement at surface 1 is

$$P_1 = \int_{-\alpha}^{+\alpha} \upsilon_r(p-\tau_{rr})rd\theta = \int_{-\alpha}^{+\alpha} f(p-\tau_{rr})d\theta$$

The pressure, p :

$$-\rho\frac{f^2}{r^3} = -\frac{\partial p}{\partial r} + \frac{\eta}{r^3}\frac{d^2f}{d\theta^2} \qquad \Rightarrow \qquad \frac{\partial p}{\partial r} = \frac{1}{r^3}\left(\rho f^2 + \eta\frac{d^2f}{d\theta^2}\right)$$

Integration gives $\qquad p = p_\infty - \dfrac{1}{2r^2}\left(\rho f^2 + \eta\dfrac{d^2f}{d\theta^2}\right)$

p_∞ : a constant

The stress component, τ_{rr} : $\tau_{rr} = 2\eta \dfrac{\partial \upsilon_r}{\partial r} = -\dfrac{2\eta f(\theta)}{r^2}$

Then, we may write P_1 in terms of $f(\theta)$,

$$P_1 = p_\infty q - \left(\frac{1}{2r_1^2}\right)\int_{-\alpha}^{\alpha}\left(\rho f^2 + \eta \frac{d^2 f}{d\theta^2} - 4\eta f\right) f\, d\theta$$

And similarly at surface 2

$$P_2 = -p_\infty q + \left(\frac{1}{2r_2^2}\right)\int_{-\alpha}^{\alpha}\left(\rho f^2 + \eta \frac{d^2 f}{d\theta^2} - 4\eta f\right) f\, d\theta$$

The net power requirement :

$$P = P_1 + P_2 = \left(\frac{1}{2r_2^2} - \frac{1}{2r_1^2}\right)\int_{-\alpha}^{\alpha}\left(\rho f^2 + \eta \frac{d^2 f}{d\theta^2} - 4\eta f\right) f\, d\theta$$

Meanwhile, $\left(\frac{2\rho}{\eta}\right) f \frac{df}{d\theta} + \frac{d^3 f}{d\theta^3} + 4\frac{df}{d\theta} = 0$

$$\Rightarrow \quad \rho \frac{df^2}{d\theta} + \eta \frac{d^3 f}{d\theta^3} + 4\eta \frac{df}{d\theta} = 0$$

Integrating once,

$$\rho f^2 + \eta \frac{d^2 f}{d\theta^2} + 4\eta f = \text{constant} = \eta f''(\alpha)$$

Substitute this into the power eq'n :

$$P = \frac{\eta}{2}\left(\frac{1}{r_2^2} - \frac{1}{r_1^2}\right)\left[f''(\alpha) q - 8\int_{-\alpha}^{\alpha} f^2(\theta)\, d\theta\right]$$

or in terms of F and ϕ,

$$P = \frac{\eta q^2}{2\alpha^3}\left(\frac{1}{r_2^2} - \frac{1}{r_1^2}\right)\left[8\alpha^2 \int_{-1}^{1} F^2(\phi)\, d\phi - F''(1)\right]$$

Define a dimensionless power requirement, $P^*(R,\alpha)$

$$P = 4\eta q^2\left[\frac{1}{(2r_2\sin\alpha)^2} - \frac{1}{(2r_1\sin\alpha)^2}\right]P^*(R,\alpha)$$

Then, $P^*(R,\alpha) = \sin^2\alpha\left[\frac{4}{\alpha}\int_{-1}^{1}F^2(\phi)\,d\phi - \frac{F''(1)}{2\alpha^3}\right]$

$R \to 0$: $$P^* = \frac{[4\alpha(1+\cos^2 2\alpha) - 4\sin 2\alpha\cos 2\alpha]\sin^2\alpha}{(\sin 2\alpha - 2\alpha\cos 2\alpha)^2}$$

$\alpha \to 0$: $$P^* = \frac{3}{4\alpha}$$

$R \to \infty$: $$P^* = \frac{|R|\sin^2\alpha}{16\alpha^3}$$: P is independent of η

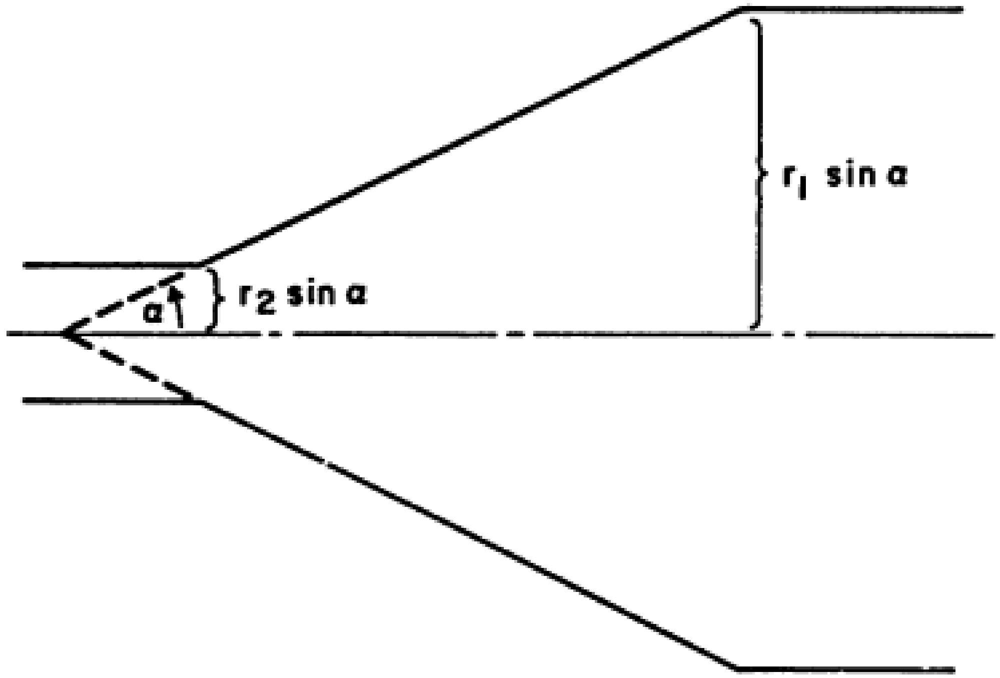

Fig. 10-7. Schematic of a finite converging flow.

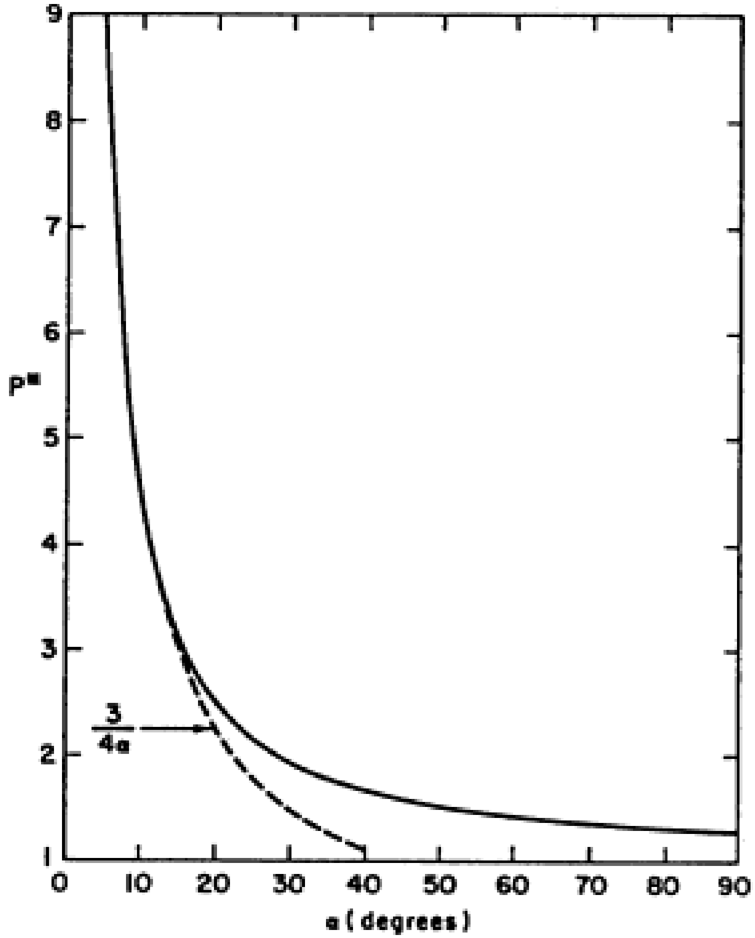

Fig. 10-8. Dimensionless power requirement as a function of angle of convergence, $R = 0$.