11. Ordering and Approximation

11.1 Introduction

Exact solutions of the Navier-Stokes eq'ns are rarely obtained.

Systematic approaches for the development of approximating solutions are required.

The logic used to obtain the simplifications is examined to develop procedures that can be applied in other flow situations.

11.2 Characteristic Quantities

Order of magnitude analysis $\Rightarrow$ important terms : retained
unimportant terms : neglected

Making each term dimensionless using characteristic quantities :

$\boldsymbol{v}/V \Rightarrow \tilde{\boldsymbol{v}}$ (dimensionless velocity vector) : order unity

a characteristic velocity

Table 11-1. Characteristic and Dimensionless Variables

$$\overline{\boldsymbol{v}} = \frac{\boldsymbol{v}}{V} \quad , \quad \overline{\boldsymbol{x}} = \frac{\boldsymbol{x}}{L} \quad , \quad \overline{\nabla} = \nabla L \quad , \quad \overline{t} = \frac{t}{T} \quad , \quad \overline{P} = \frac{P}{\Pi}$$

The continuity eq'n :

$$\nabla \cdot \boldsymbol{v} = 0 \quad \Rightarrow \quad \frac{V}{L}\widetilde{\nabla} \cdot \boldsymbol{v} = 0 \quad \text{or} \quad \widetilde{\nabla} \cdot \boldsymbol{v} = 0$$

The Navier-Stokes eq'ns :

$$\rho \frac{\partial \boldsymbol{v}}{\partial t} + \rho \boldsymbol{v} \cdot \nabla \boldsymbol{v} = -\nabla P + \eta \nabla^2 \boldsymbol{v}$$

nondimensionalization

$$\left(\frac{\rho V}{T}\right)\frac{\partial \tilde{\boldsymbol{v}}}{\partial \tilde{t}} + \left(\frac{\rho V^2}{L}\right)\tilde{\boldsymbol{v}} \cdot \widetilde{\nabla}\tilde{\boldsymbol{v}} = -\left(\frac{\Pi}{L}\right)\widetilde{\nabla}\tilde{P} + \left(\frac{\eta V}{L^2}\right)\widetilde{\nabla}^2\tilde{\boldsymbol{v}}$$

or $$\left(\frac{\rho L^2}{\eta T}\right)\frac{\partial \tilde{\boldsymbol{v}}}{\partial \tilde{t}} + \left(\frac{\rho V L}{\eta}\right)\tilde{\boldsymbol{v}} \cdot \widetilde{\nabla}\tilde{\boldsymbol{v}} = -\left(\frac{\Pi L}{\eta V}\right)\widetilde{\nabla}\tilde{P} + \widetilde{\nabla}^2\tilde{\boldsymbol{v}}$$

: Reynolds number, Re

$$\left(\frac{\rho L^2}{\eta T}\right)\frac{\partial \tilde{\boldsymbol{v}}}{\partial \tilde{t}} + \mathrm{Re}\,\tilde{\boldsymbol{v}} \cdot \widetilde{\nabla}\tilde{\boldsymbol{v}} = -\left(\frac{\Pi L}{\eta V}\right)\widetilde{\nabla}\tilde{P} + \widetilde{\nabla}^2\tilde{\boldsymbol{v}}$$

or $\quad \mathrm{Re}(\mathrm{Sr}^{-1}\dfrac{\partial\, \tilde{\boldsymbol{\upsilon}}}{\partial\, \tilde{t}} + \tilde{\boldsymbol{\upsilon}} \cdot \tilde{\nabla}\tilde{\boldsymbol{\upsilon}}) = -(\dfrac{\Pi L}{\eta V})\tilde{\nabla}\tilde{P} + \tilde{\nabla}^2\tilde{\boldsymbol{\upsilon}}$

where $\mathrm{Sr} = \dfrac{VT}{L}$: Strouhal number

11.3 Characteristic Pressure

Inertially dominated flows : $\Pi = \rho V^2$

Then, N-S eq'ns becomes

$$\mathrm{Re}(\mathrm{Sr}^{-1}\frac{\partial\, \tilde{\boldsymbol{\upsilon}}}{\partial\, \tilde{t}} + \tilde{\boldsymbol{\upsilon}} \cdot \tilde{\nabla}\tilde{\boldsymbol{\upsilon}} + \tilde{\nabla}\tilde{P}) = \tilde{\nabla}^2\tilde{\boldsymbol{\upsilon}}$$

Viscous dominated flows : $\Pi = \dfrac{\eta V}{L}$

$$\mathrm{Re}(\mathrm{Sr}^{-1}\frac{\partial\, \tilde{\boldsymbol{\upsilon}}}{\partial\, \tilde{t}} + \tilde{\boldsymbol{\upsilon}} \cdot \tilde{\nabla}\tilde{\boldsymbol{\upsilon}}) = -\tilde{\nabla}\tilde{P} + \tilde{\nabla}^2\tilde{\boldsymbol{\upsilon}}$$

11.4 Characteristic Time

When the characteristic time is given by the time required for a fluid element to move through a characteristic distance,

characteristic time = flow time : $T = \frac{L}{V}$ $\Rightarrow$ Sr = 1

$$\mathrm{Re}\left(\frac{\partial \tilde{\boldsymbol{v}}}{\partial \tilde{t}} + \tilde{\boldsymbol{v}} \cdot \tilde{\nabla} \tilde{\boldsymbol{v}}\right) = -\tilde{\nabla}\tilde{P} + \tilde{\nabla}^2 \tilde{\boldsymbol{v}}$$

11.5 Ordering Arguments

All dimensionless quantities are of order unity.
Changes in dimensionless quantities taking place over dimensionless length scales are of order unity. $\Rightarrow$ all dimensionless derivatives are of order unity.

The relative importance of each term is therefore determined by the magnitude of the dimensionless group.

(Example)

Consider the case of a long cylinder of radius R immersed in an infinite body of fluid. The cylinder oscillates about its axis with frequency ω : $v_\theta = V \sin \omega t$ at $r = R$

$$\mathrm{Re} = \frac{RV\rho}{\eta} \quad , \quad \mathrm{Sr} = \frac{V}{R\omega}$$

$$* \quad \omega \ll \frac{V}{R} \; , \quad \mathrm{Sr} \gg 1 : \qquad \left| \mathrm{Sr}^{-1} \frac{\partial \tilde{\boldsymbol{v}}}{\partial \tilde{t}} \right| \ll | \tilde{\boldsymbol{v}} \cdot \tilde{\nabla} \tilde{\boldsymbol{v}} |$$

$$* \quad \omega \gg \frac{V}{R} \; , \quad \mathrm{Sr} \ll 1 : \qquad \left| \mathrm{Sr}^{-1} \frac{\partial \tilde{\boldsymbol{v}}}{\partial \tilde{t}} \right| \gg | \tilde{\boldsymbol{v}} \cdot \tilde{\nabla} \tilde{\boldsymbol{v}} |$$

Then,

$$* \quad \mathrm{Sr} \gg 1 : \quad \mathrm{Re}\, \tilde{\boldsymbol{v}} \cdot \tilde{\boldsymbol{\nabla}} \tilde{\boldsymbol{v}} = -\tilde{\boldsymbol{\nabla}} \tilde{P} + \widetilde{\boldsymbol{\nabla}^2} \tilde{\boldsymbol{v}}$$

pseudo-steady-state eq'n

$$* \quad \mathrm{Sr} \ll 1 : \quad \frac{\mathrm{Re}}{\mathrm{Sr}} \frac{\partial \tilde{\boldsymbol{v}}}{\partial \tilde{t}} = -\tilde{\boldsymbol{\nabla}} \tilde{P} + \widetilde{\boldsymbol{\nabla}^2} \tilde{\boldsymbol{v}}$$

linear eq'n

When Re is small,

$$* \quad \mathrm{Sr} \gg 1 \text{ and } \mathrm{Re} \ll 1 : \quad 0 = -\tilde{\boldsymbol{\nabla}} \tilde{P} + \widetilde{\boldsymbol{\nabla}^2} \tilde{\boldsymbol{v}}$$

$$\left(\omega \ll \frac{V}{R} \ll \frac{\eta}{\rho R^2} \right)$$

$$* \quad \mathrm{Sr} \ll 1 \text{ and } \mathrm{Re} \ll \mathrm{Sr} : \quad 0 = -\tilde{\boldsymbol{\nabla}} \tilde{P} + \widetilde{\boldsymbol{\nabla}^2} \tilde{\boldsymbol{v}}$$

$$\left(\frac{V}{R} \ll \omega \ll \frac{\eta}{\rho R^2} \right)$$

11.6 Solution Logic

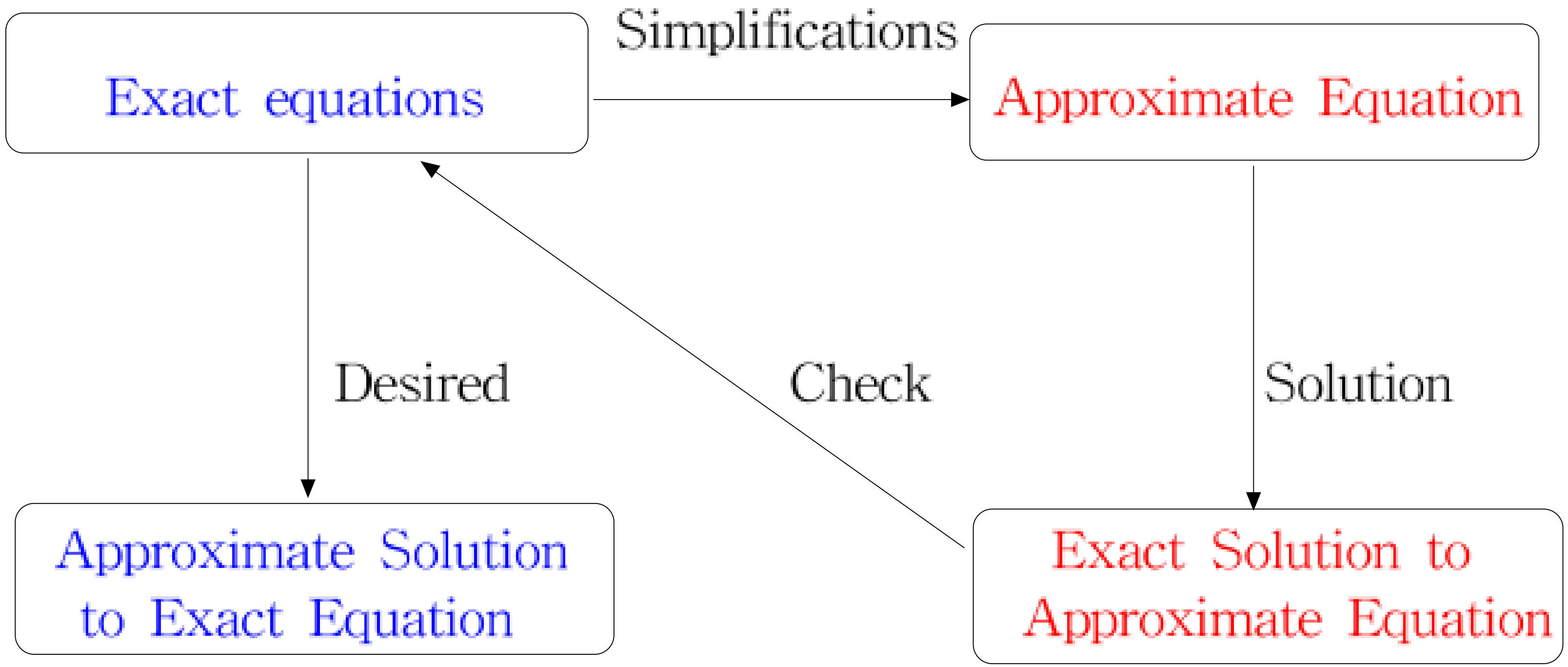

Fig. 11-1. Logic of the process for obtaining approximate solutions.

(Counterexample)

$$0.01x + y = 0.1$$

$$x + 101y = 11$$

Assume that $O(x) \approx O(y)$. $\Rightarrow$ neglect the first term in the first eq′n

$$y = 0.1$$

$$x + 101y = 11$$

$\Rightarrow$ $y = 0.1$ and $x = 0.9$

the neglected term : $0.01x = 0.009$ looks perfect

But, the exact solution : $x = -90$ and $y = 1$

11.7 Inviscid Limit

Inertially dominated flows :

$$\mathrm{Sr}^{-1}\frac{\partial \tilde{\boldsymbol{v}}}{\partial \tilde{t}}+\tilde{\boldsymbol{v}}\cdot\tilde{\nabla}\tilde{\boldsymbol{v}}+\tilde{\nabla}\tilde{P}=\mathrm{Re}^{-1}\widetilde{\nabla^2}\tilde{\boldsymbol{v}}$$

$$\mathrm{Re}\rightarrow\infty\ :\quad \mathrm{Sr}^{-1}\frac{\partial \tilde{\boldsymbol{v}}}{\partial \tilde{t}}+\tilde{\boldsymbol{v}}\cdot\tilde{\nabla}\tilde{\boldsymbol{v}}+\tilde{\nabla}\tilde{P}=0$$

: inviscid fluid flow ($\eta = 0$)

: singular, second derivative has been lost

one boundary condition cannot be satisfied.

(no-slip condition)