12. Creeping Flow

12.1 Introduction

When a flow has a single characteristic length L and characteristic velocity V, and have no characteristic time other than $T = L/V$,

$$\mathrm{Re}\left(\frac{\partial \tilde{\boldsymbol{v}}}{\partial \tilde{t}} + \tilde{\boldsymbol{v}} \cdot \tilde{\nabla} \tilde{\boldsymbol{v}}\right) = -\tilde{\nabla}\tilde{P} + \widetilde{\nabla^2}\tilde{\boldsymbol{v}} \quad , \qquad \mathrm{Re} = \frac{LV\rho}{\eta}$$

When Re is very small, we may ignore the inertial terms and write the creeping flow approximation to the N-S eq'ns,

$$\mathrm{Re} \ll 1 : \quad 0 = -\tilde{\nabla}\tilde{P} + \widetilde{\nabla^2}\tilde{\boldsymbol{v}}$$

or in dimensional form,

$$\mathrm{Re} \ll 1 : \quad 0 = -\nabla P + \eta \nabla^2 \boldsymbol{v}$$

It applies to flows that are "slow" in the sense of having a small Re

by a very small characteristic velocity, V, or

a very small characteristic length, L, or

a very large viscosity, η.

The creeping flow approximation is nearly always applicable for polymer melt flows due to the large viscosity of molten polymers.

Mathematically it is a reduction of the nonlinear N-S eq′ns to a set of linear eq′ns.

12.2 Flow Between Rotating Disks

Consider the flow between rotating disks shown below.
We wish to determine the velocity field and the torque to turn the moving disk.

$R/H >> 1$

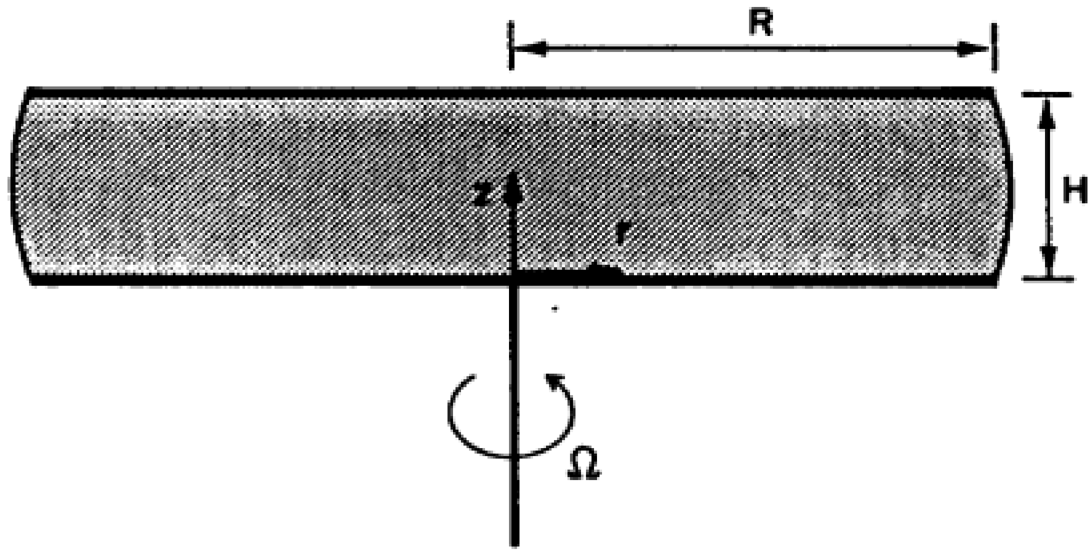


Fig. 12-1. Schematic of flow between rotating disk.

Kinematic assumption : (using cylindrical coordinates)

$$\frac{\partial}{\partial \theta} = 0 \quad \text{(no variation in the angular direction)}$$

Continuity and N-S eq'ns :

$$\frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{\partial v_z}{\partial z} = 0$$

$$0 = -\frac{\partial P}{\partial r} + \eta\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r} r v_r\right) + \frac{\partial^2 v_r}{\partial z^2}\right]$$

$$0 = \eta\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r} r v_\theta\right) + \frac{\partial^2 v_\theta}{\partial z^2}\right]$$

$$0 = -\frac{\partial P}{\partial z} + \eta\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{\partial^2 v_z}{\partial z^2}\right]$$

The B.C. :

$$z = 0 : \quad v_r = v_z = 0 \quad v_\theta = r\Omega$$

$$z = H : \quad v_r = v_z = v_\theta = 0$$

The eq'n for v_θ is uncoupled.

$$0 = \eta[\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial}{\partial r} r v_\theta) + \frac{\partial^2 v_\theta}{\partial z^2}]$$

The boundary condition for v_θ suggests us to look for a solution in the form of $v_\theta = r f(z)$ with $f(0) = \Omega \quad f(H) = 0$.

Then, we obtain $\frac{d^2 f}{dz^2} = 0 \quad \Rightarrow \quad f(z) = \Omega(1 - \frac{z}{H})$

Terefore, $v_\theta = r\Omega(1 - \frac{z}{H})$

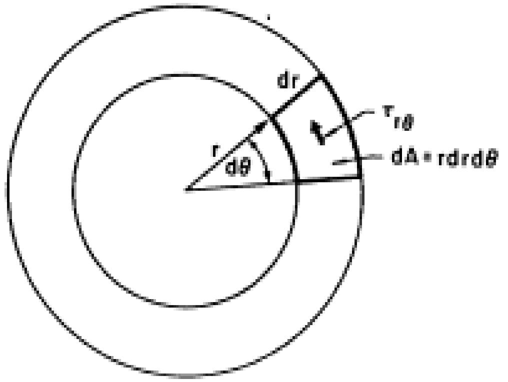
The torque :

The shear stress at the lower plate

$$\tau_{z\theta} = \eta\left(\frac{\partial v_\theta}{\partial z} + \frac{\partial v_z}{\partial \theta}\right) = -\frac{\eta r \Omega}{H}$$

The differential force on the sector of area $r\,dr\,d\theta$ is

$$dF = \tau_{z\theta} r\,dr\,d\theta = -\frac{\eta\Omega}{H} r^2\,dr\,d\theta$$



The differential torque is

$$dG = r\,dF = -\frac{\eta\Omega}{H} r^3\,dr\,d\theta$$

The total torque is

$$G = \int dG = -\frac{\eta\Omega}{H}\int_0^R \int_0^{2\pi} r^3\,dr\,d\theta = -\frac{\pi\eta\Omega R^4}{2H}$$

* Parallel-disk Viscometer : η from G

Checking the creeping flow condition :

$$\text{Inertial terms} \sim \frac{\rho v_\theta^2}{r} \sim \rho r \Omega^2$$

$$\text{Viscous terms} \sim \eta \frac{\partial^2 v_\theta}{\partial z^2} \sim \frac{\eta r \Omega}{H^2}$$

$$\frac{\text{inertial terms}}{\text{viscous terms}} \sim \frac{\rho r \Omega^2}{\eta r \Omega / H^2} \sim \frac{H^2 \Omega \rho}{\eta} \ll 1$$

: can be easily satisfied

12.3 Flow Around A Sphere

Consider the flow around a sphere to derive Stokes′ law (in Ch. 4).

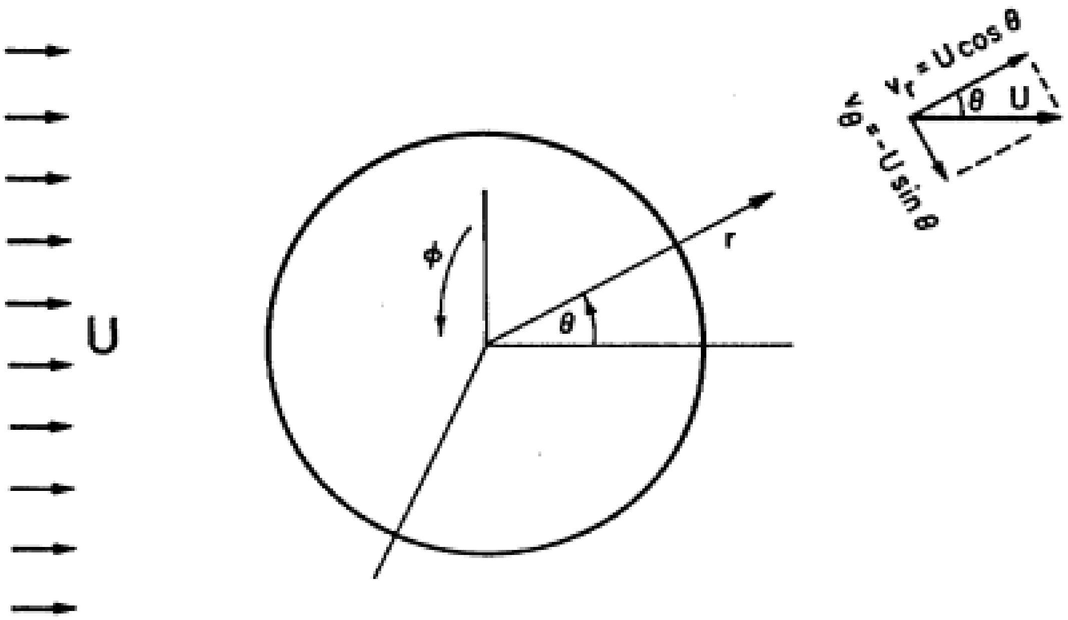


Fig. 12-3. Schematic of uniform flow past a sphere.

Kinematic assumption : (using spherical coordinates)

$$\frac{\partial}{\partial \phi} = 0, \qquad v_\phi = 0$$

Continuity eq'n :

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(v_\theta \sin\theta) = 0$$

N-S eq'ns :

$$0 = -\frac{\partial P}{\partial r} + \eta\Big[\frac{1}{r^2}\frac{\partial}{\partial r}\Big(r^2\frac{\partial v_r}{\partial r}\Big) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\Big(\sin\theta\frac{\partial v_r}{\partial \theta}\Big) - \frac{2v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} - \frac{2}{r^2}v_\theta\cot\theta\Big]$$

$$0 = -\frac{1}{r}\frac{\partial P}{\partial \theta} + \eta\Big[\frac{1}{r^2}\frac{\partial}{\partial r}\Big(r^2\frac{\partial v_\theta}{\partial r}\Big) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\Big(\sin\theta\frac{\partial v_\theta}{\partial \theta}\Big) + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r^2\sin^2\theta}\Big]$$

The B.C. :

$$r = R: \quad v_r = v_\theta = 0$$

$$r \to \infty: \quad v_r = U\cos\theta \qquad v_\theta = -U\sin\theta$$

Again the B.C. suggests us to look for the solution in the form of

$$v_r = A(r)\cos\theta \quad , \qquad v_\theta = B(r)\sin\theta$$

with $r = R: \quad A(R) = B(R) = 0$

$r \to \infty: \quad A(\infty) = U \qquad B(\infty) = -U$

* 해를 구하는 중간과정 생략 : 식 (12.26) - (12.33) 참조 요망.

The solution is

$$v_r = U\left[1 - \frac{3}{2}\frac{R}{r} + \frac{1}{2}\left(\frac{R}{r}\right)^3\right]\cos\theta$$

$$v_\theta = -U\left[1 - \frac{3}{4}\frac{R}{r} - \frac{1}{4}\left(\frac{R}{r}\right)^3\right]\sin\theta$$

$$P = P_0 - \frac{3\eta U}{2R}\left(\frac{R}{r}\right)^2\cos\theta$$

Form and friction drag :

* pressure profile

$$P = p + \rho g h = P_0 - \frac{3\eta U}{2R}\left(\frac{R}{r}\right)^2 \cos\theta$$

$$h = r\cos(\theta - \alpha) = r(\cos\alpha\cos\theta + \sin\alpha\sin\theta)$$

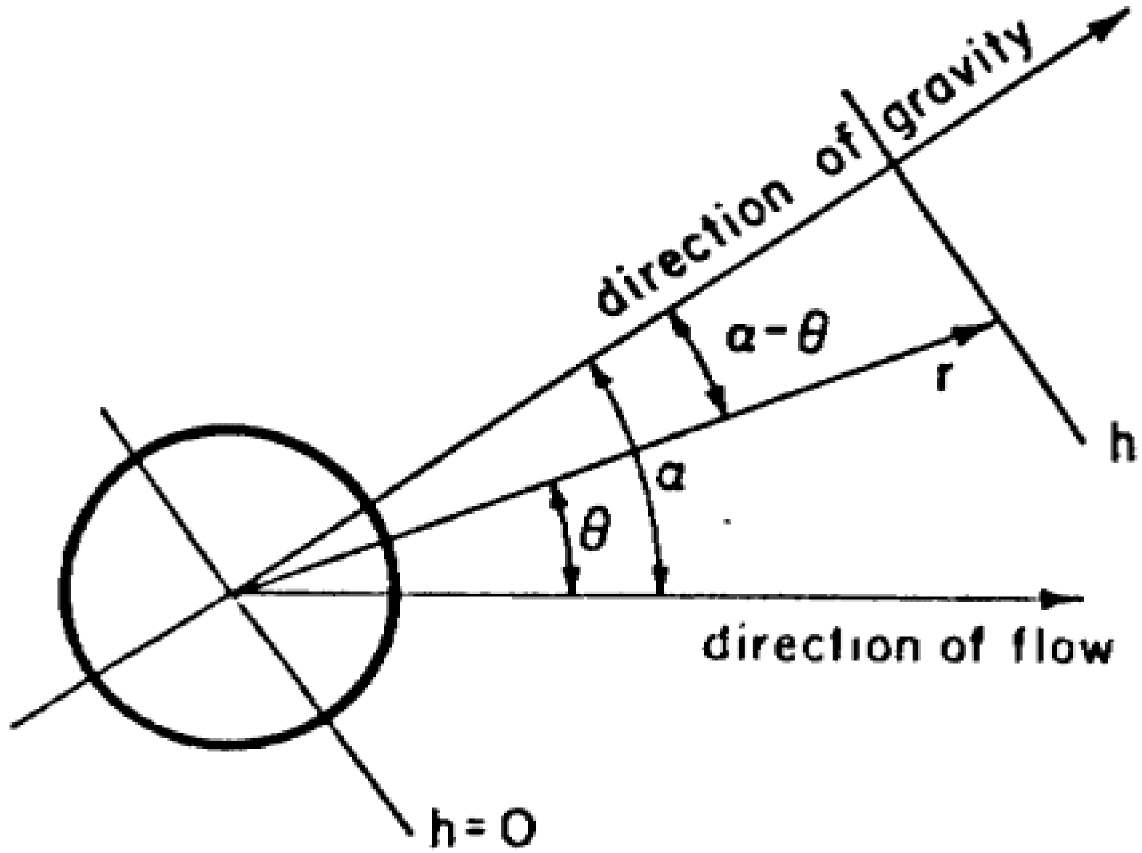
$$\therefore\ p = P_0 - \rho g r(\cos\alpha\cos\theta + \sin\alpha\sin\theta) - \frac{3\eta U}{2R}\left(\frac{R}{r}\right)^2 \cos\theta$$

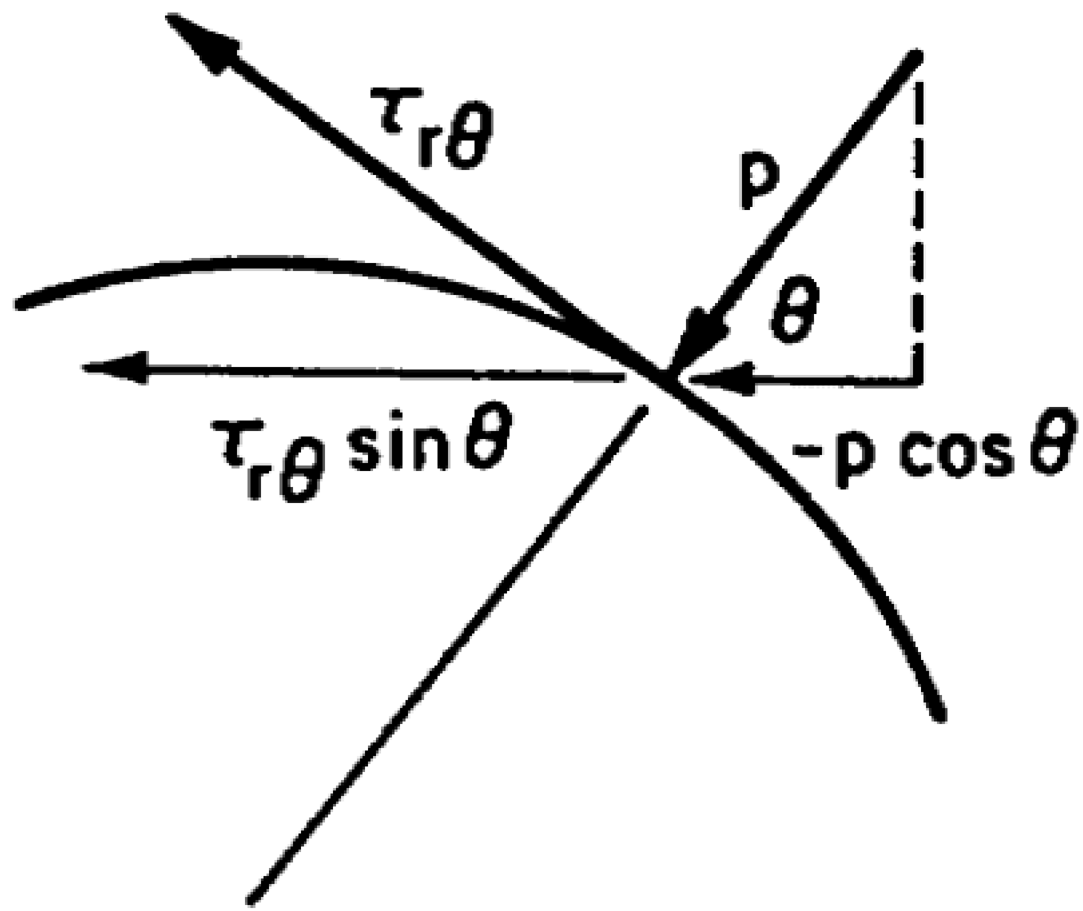
$r = R$: $$p = P_0 - \rho g R(\cos\alpha\cos\theta + \sin\alpha\sin\theta) - \frac{3\eta U}{2R}\cos\theta$$

* shear stress

$$\tau_{r\theta} = \eta\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right]$$

$r = R$: $$\tau_{r\theta} = -\frac{3\eta U}{2R}\sin\theta$$


direction of gravity
α-θ
r
h
α
θ
direction of flow
h=O



$\tau_{r\theta}$
p
θ
$\tau_{r\theta} \sin\theta$
-p cos θ

* net force on the sphere in the flow direction

$$F = 2\int_{\phi=0}^{\phi=2\pi}\int_{\theta=0}^{\theta=\pi}(-p\cos\theta-\tau_{r\theta}\sin\theta)R^2\sin\theta\,d\theta\,d\phi$$

$$= 2\int_{\phi=0}^{\phi=2\pi}\int_{\theta=0}^{\theta=\pi}[-P_0+\rho gR(\cos\alpha\cos\theta+\sin\alpha\sin\theta)$$

$$+\frac{3\eta U}{2R}\cos\theta]R^2\sin\theta\cos\theta\,d\theta\,d\phi$$

$$+2\int_{\phi=0}^{\phi=2\pi}\int_{\theta=0}^{\theta=\pi}\frac{3\eta U}{2R}\sin^2\theta R^2\sin\theta\,d\theta\,d\phi$$

$$= \frac{4\pi}{3}\rho gR^3\cos\alpha+2\pi\eta RU+4\pi\eta RU$$

buoyancy force, form drag, friction drag

* total drag force is

$F_D = 6\pi\eta RU = 3\pi\eta DU$: Stokes' law

Checking the creeping flow condition

inertial : $\rho \upsilon_r \frac{\partial \upsilon_r}{\partial r} \sim \frac{3\rho U^2 R}{2r^2}$

viscous : $\eta \frac{1}{r^2} \frac{\partial}{\partial r}(r^2 \frac{\partial \upsilon_r}{\partial r}) \sim \frac{3\eta U R^3}{r^5}$

$$\frac{\text{inertial}}{\text{viscous}} \sim \frac{3\rho U^2 R/2r^2}{3\eta U R^3/r^5} = \frac{1}{2}\frac{RU\rho}{\eta}\left(\frac{r}{R}\right)^3 = \frac{1}{4}\mathrm{Re}\left(\frac{r}{R}\right)^3$$

12.4 Squeeze Film

Consider the flow between two parallel disks, one of which (upper one) is moving towards the other under the imposed force. Compute the relationship between the imposed force (F), the speed (V) and the spacing of the plates (H), and time (t).

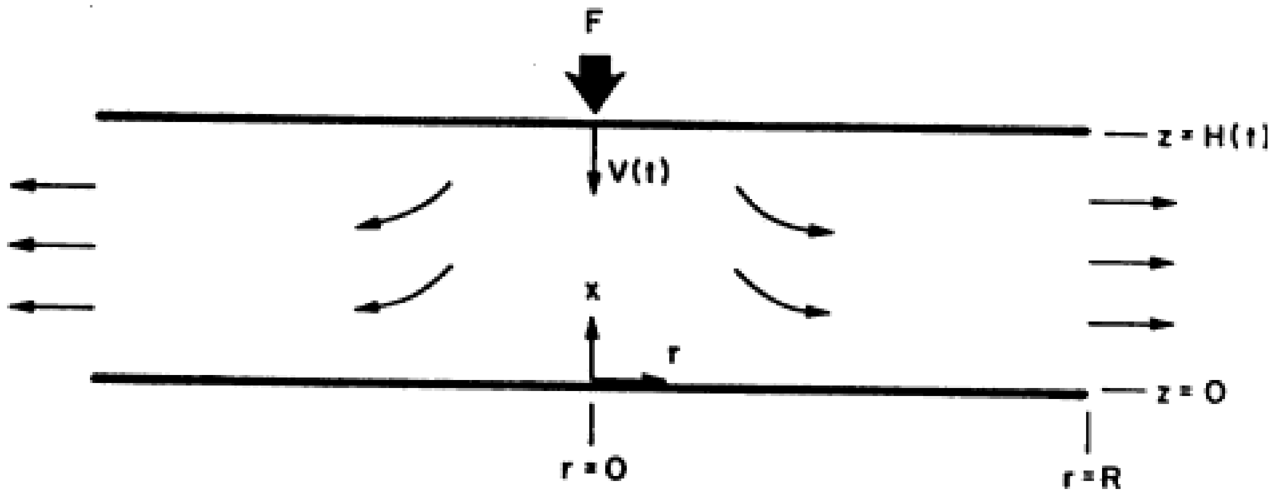


Fig. 12-6. Schematic of a squeeze film.

The Reynolds number :

$$\mathrm{Re} = \frac{H_0 V_m \rho}{\eta}$$

H_0 : The initial spacing of the disks

V_m : The maximum velocity of the upper plate

The characteristic time : $T = \dfrac{H_0}{V_m}$

Therefore, in the limit $\mathrm{Re} \rightarrow 0$,

$$0 = -\nabla P + \eta \nabla^2 \boldsymbol{v}$$

: The creeping flow limit for the transient flow introduces a pseudosteady-state assumption

Kinematic assumption : (using cylindrical coordinates)

$$\upsilon_\theta = 0, \quad \frac{\partial}{\partial \theta} = 0 \quad \Rightarrow \quad \upsilon_r = \upsilon_r(r,z), \; \upsilon_z = \upsilon_z(r,z)$$

The continuity eq'n :

$$\frac{1}{r}\frac{\partial}{\partial r}(r\upsilon_r) + \frac{\partial \upsilon_z}{\partial z} = 0$$

The N-S eq'ns :

$$0 = -\frac{\partial P}{\partial r} + \eta\left[\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}(r\upsilon_r) + \frac{\partial^2 \upsilon_r}{\partial z^2}\right]$$

$$0 = -\frac{\partial P}{\partial z} + \eta\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \upsilon_z}{\partial r}\right) + \frac{\partial^2 \upsilon_z}{\partial z^2}\right]$$

The B.C. :

$$z = 0: \quad \upsilon_z = \upsilon_r = 0$$

$$z = H(t): \quad \upsilon_z = -V(t) \quad \upsilon_r = 0$$

The axial velocity (υ_z) is independent of r at the upper and lower plates. ⇒ assume that $\upsilon_z = \Phi(z)$: parallel squeezing assumption

From the continuity eq'n, we then have

$$\frac{1}{r}\frac{\partial}{\partial r}(r\upsilon_r) = -\frac{d\Phi(z)}{dz}$$

or, after integrating

$$\upsilon_r = -\frac{r}{2}\frac{d\Phi}{dz} + \frac{\text{constant}}{r} = -\frac{r}{2}\frac{d\Phi}{dz}$$

Using the above υ_r and υ_z, N-S eq'ns become

$$0 = -\frac{\partial P}{\partial r} - \frac{1}{2}\eta r\frac{d^3\Phi}{dz^3}$$

$$0 = -\frac{\partial P}{\partial z} + \eta\frac{d^2\Phi}{dz^2}$$

Eliminating the pressure by cross-differentiation,

$$\frac{d^4\Phi}{dz^4} = 0$$

The B.C. in terms of $\Phi(z)$ are

$$z = 0 : \quad \Phi = \frac{d\Phi}{dz} = 0$$

$$z = H(t) : \quad \Phi = -V(t) \quad \frac{d\Phi}{dz} = 0$$

The solutions are

$$\upsilon_z = \Phi(z) = -3V(t)\left[\frac{z}{H(t)}\right]^2\left[1 - \frac{2}{3}\frac{z}{H(t)}\right]$$

$$\upsilon_r = -\frac{r}{2}\frac{d\Phi}{dz} = \frac{3rzV(t)}{H^2(t)}\left[1 - \frac{z}{H(t)}\right] \quad , \quad V(t) = -\frac{dH(t)}{dt}$$

The pressure P :

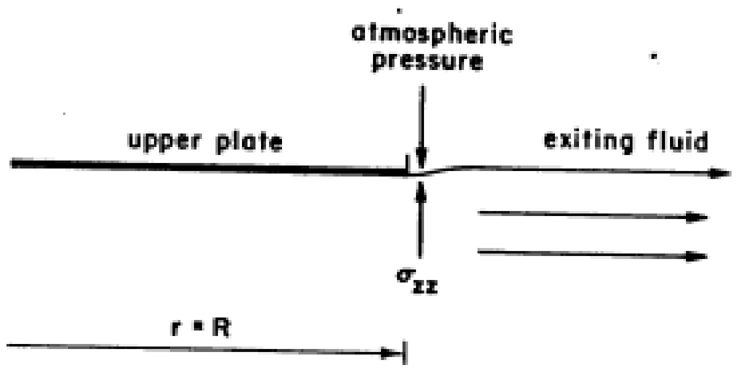
$$\frac{\partial P}{\partial r} = -\frac{1}{2}\eta r \frac{d^3\phi}{dz^3} = -\frac{6\eta V r}{H^3}$$

$$\frac{\partial P}{\partial z} = \eta \frac{d^2\phi}{dz^2}$$

$$\Rightarrow \quad P = P_0 + \frac{3\eta V}{H}\left[2\frac{z}{H}\left(\frac{z}{H}-1\right)-\frac{r^2}{H^2}\right]$$

P_0 is determined using the condition of

$$z = H,\ r = R : \quad \sigma_{zz} = -p + 2\eta\frac{\partial \upsilon_z}{\partial z} = 0$$



$$P_0 = \frac{3\eta V R^2}{H^3} \quad \Rightarrow \quad P(r,z) = \frac{3\eta V}{H}\left[2\frac{z}{H}\left(\frac{z}{H}-1\right)+\frac{R^2-r^2}{H^2}\right]$$

The total force on the upper plate :

$$F = \int_0^R 2\pi r(-\sigma_{zz})_{z=H}\, dr$$

$$= \int_0^R 2\pi r P(r,H)\, dr = \frac{3\pi\eta V R^4}{2H^3}$$

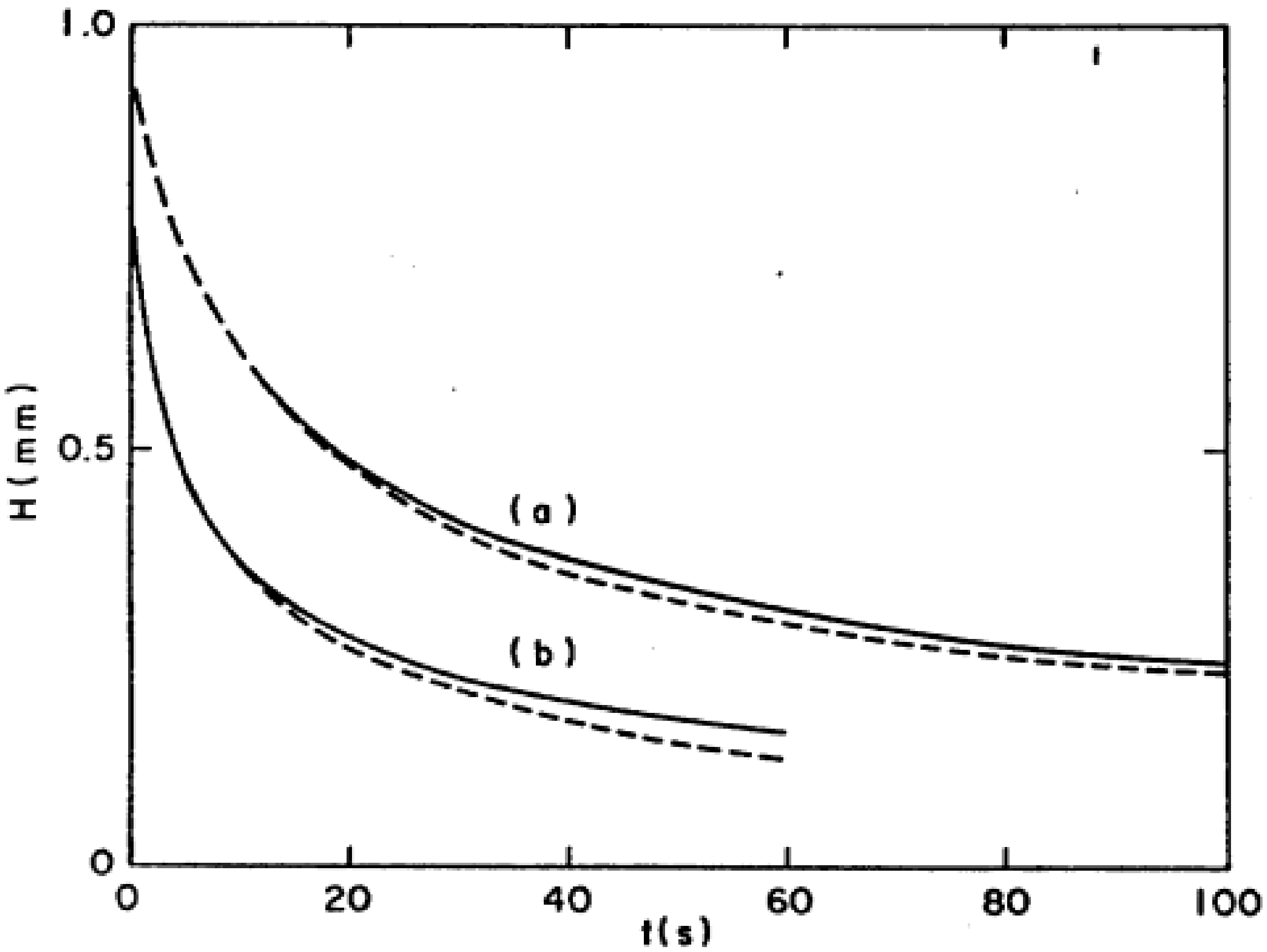
When the force is constant,

$$-V = \frac{dH}{dt} = -\left(\frac{2F}{3\pi\eta R^4}\right)H^3$$

$$\Rightarrow \quad H(t) = \left(\frac{1}{H_0^2}+\frac{4Ft}{3\pi\eta R^4}\right)^{-1/2}$$: Stefan eq'n

The range of applicability :

$$\mathrm{Re} = \frac{HV\rho}{\eta} = \frac{2\rho F H^4}{3\pi\eta^2 R^4} << 1 \quad \text{or} \quad F << \frac{\eta^2}{\rho}\left(\frac{R}{H}\right)^4$$



1.0
0.5
0
H (mm)
0
20
40
60
80
100
t(s)
(a)
(b)