3. Pipe Flow

3.1 Introduction

3.2 Dimensional Analysis

A physical process involves a relation between V variables,

$$x_1 = \text{function of}(x_2, x_3, \cdots, x_V)$$

If the V variables require D dimensions (L, M, θ, etc.), then the variables can be combined into $G = V - D$ independent dimensionless groups of the form

$$N_1 = x_1^{a_1} x_2^{a_2} \cdots x_V^{a_V}, \quad N_2 = x_1^{b_1} x_2^{b_2} \cdots x_V^{b_V}, \text{ and so on.}$$

One then obtains

$$N_1 = \text{function of}(N_2, N_3, \cdot\cdot\cdot, N_G)$$

: **Buckingham pi theorem**

It reduces the number of variables from V to G.

It indicates what groupings of physical variables affect the process.

It may be possible to limit the scope of an experimental program and to design experiments on one physical scale that are applicable to the process on an entirely different physical scale.

3.3 Smooth Pipe Flow of Newtonian Fluid

Dimensionless Groups:

Variables and dimensions for pipe flow

Variables	Dimensions	
Diameter, D	length	L
Length, L	length	L
Pressure Drop, $\lvert\Delta P\rvert$	force/area	$ML^{-1}\theta^{-2}$
Mean Velocity, υ	length/time	$L\theta^{-1}$
Density, ρ	mass/volume	ML^{-3}
Viscosity, η	(force/area)xtime	$ML^{-1}\theta^{-1}$

6 variables, 3 dimensions; 3 independent dimensionless groups

L/D is dimensionless.

$|\Delta P|$ has the same dimensions as a stress.

$\eta v/D$ has dimensions of stress.

$\frac{1}{2}\rho v^2$ is kinetic energy per unit volume and has the same dimensions as a stress or a pressure.

Thus, $(\rho v^2)/(\eta v/D)$ or $Dv\rho/\eta$ is dimensionless.

$|\Delta p|/\rho v^2$ is also dimensionless.

The three most convenient groups to use are

$$\frac{|\Delta p|}{\rho v^2} \qquad \frac{Dv\rho}{\eta} \qquad \frac{L}{D}$$

One can then write

$$\frac{|\Delta p|}{\rho v^2} = \text{function of } \left(\frac{Dv\rho}{\eta} , \ \frac{L}{D} \right)$$

If L is replaced by $2L$, $|\Delta P|$ is doubled.

Therefore,

$$\frac{|\Delta p|}{\rho v^2} = \frac{L}{D} \text{ x function only of } \frac{Dv\rho}{\eta}$$

Let us introduce two dimensionless groups,

$$\frac{Dv\rho}{\eta} \ : \text{Reynolds number (Re)}$$

$$\frac{|\Delta p|}{2\rho v^2}\frac{D}{L} \ : \text{(Fanning) friction factor } (f\ )$$

Then, $f = f(\text{Re})$

The **friction factor** is a **unique function of the Reynolc** **number** for smooth pipe flow of all incompressible Newtonian fluids.

Friction Factor - Reynolds Number Data:

The form of the relation, $f = f(\mathrm{Re})$, must be determined from experiment or from a more fundamental analysis of the process.

Figure 3-1. f vs. Re for incompressible Newtonian fluids; data cover a broad range of viscosities, densities, and pipe diameters.

Figure 3-2. Dye stream experiment.

Laminar flow : Re < 2100

$$f = \frac{16}{Re} \quad \text{or} \quad Q = \frac{\pi}{128}\frac{|\Delta p|D^4}{L\eta}$$: Hagen–Poiseuille eq′n

Transition regime : 2100 < Re < 4000

Turbulent flow : Re > 4000

Blasius eq′n $f = 0.079\,\mathrm{Re}^{-\frac{1}{4}}$ $(4000 \leq \mathrm{Re} \leq 10^5)$

$$\text{or} \quad Q = 2.26\left(\frac{|\Delta p|}{L}\right)^{\frac{4}{7}}(\rho^3\eta)^{-\frac{1}{7}}D^{\frac{19}{7}}$$

von Karman–Nikuradse eq′n

$$\frac{1}{\sqrt{f}} = 4.0\log_{10}(\mathrm{Re}\sqrt{f}) - 0.4 \qquad (\mathrm{Re} \geq 4000)$$

Capillary Viscometry: (advanced treatment in the section 19.6)

Hagen-Poiseuille eq'n :

$$Q = \frac{\pi}{128}\frac{|\Delta p| D^4}{L\eta}$$

or $\eta = \dfrac{\pi}{128}\dfrac{|\Delta p| D^4}{LQ}$: used in the capillary viscometry

Ex. 3.1 $|\Delta p|$ = 200 Pa, L = 1 m, D = 10 mm, Q = 60 mm^3/s

$$\eta = \frac{\pi}{128}\frac{200\,\text{Pa}}{1\,\text{m}}\frac{(10^{-2}\,\text{m})^4}{60 \times 10^{-9}\,\text{m}^3/\text{s}} = 0.82\,\text{Pa}\cdot\text{s}$$

check if Re < 2100 : assume that the density of the fluid is of order 10^3 kg/m^3, then,

$$\text{Re} = \frac{Dv\rho}{\eta} = \frac{4Q\rho}{\pi D\eta} = \frac{4 \times (60 \times 10^{-9}\,\text{m}^3/\text{s})(10^3\,\text{kg/m}^3)}{\pi \times (10^{-2}\,\text{m})(0.82\,\text{Pa}\cdot\text{s})} \approx 10^{-2}$$

End Effects:

There is a short region near the entrance and the exit of the pipe where the flow is adjusting.

The entry length, L_e, is given approximately by

$$\frac{L_e}{D} \approx 0.59 + 0.055\,\mathrm{Re} \qquad (\mathrm{Re} < 2100)$$

$$\frac{L_e}{D} \approx 40 \qquad (\mathrm{Re} > 2100)$$

Important in the capillary viscometers for measuring the viscosity of very viscous liquids.

Physical Meaning of Re and f :

The forces acting on a fluid element

Inertial force $F_I = \frac{mv}{\Delta t} = \frac{mv^2}{\Delta l} \approx \frac{\rho A d v^2}{\Delta l} \approx \rho v^2 d^2$

Viscous shear force $F_V = \frac{\eta v}{\Delta l} A \approx \eta v d$

External force $F_E = |\Delta p| A \approx |\Delta p| d^2$

Re : the ratio of the inertial force to the viscous shear force

$$\frac{F_I}{F_V} \approx \frac{\rho v^2 d^2}{\eta v d} = \frac{\rho v d}{\eta}$$

f : The ratio of the net imposed external force to the inertial force

$$\frac{F_E}{F_I} \approx \frac{|\Delta p| d^2}{\rho d^2 v^2} = \frac{|\Delta p|}{\rho v^2}$$

Laminar flow ⇒ constant velocity ⇒ no momentum change

⇒ no inertial force

$$f = f(Re) \quad \Rightarrow \quad |\Delta p| D^2 = \text{ proportional to } F_I f\left(\frac{F_I}{F_V}\right)$$

$$\therefore \; f\left(\frac{F_I}{F_V}\right) \propto \left(\frac{F_I}{F_V}\right)^{-1} \propto (\mathrm{Re})^{-1} \quad \text{or} \quad f\,\mathrm{Re} = \text{constant}$$

3.4 Power

Power Input:

The external force acting on the fluid in the pipe is $|\Delta p|\pi D^2/4$.

The work done to move the fluid in the pipe a distance Δl is $|\Delta p|(\pi D^2/4)\Delta l$.

This takes place over a time Δt, so the rate of doing work, i.e., the power input is

power input :
$$\begin{aligned} P &= |\Delta p|(\pi D^2/4)(\Delta l/\Delta t) \\ &= |\Delta p|(\pi D^2/4)v \\ &= Q|\Delta p| \end{aligned}$$

Dissipation:

Power input

⇒ the rate at which work is being done on the flowing system

⇒ increase in the energy of the system

⇒ increase in the fluid temperature

The maximum increase in the fluid temperature in an adiabatic pipe is obtained from the energy balance

$$P = Q|\Delta p| = \rho c_v Q \Delta T \qquad \text{or} \qquad \Delta T = \frac{|\Delta p|}{\rho c_v}$$

This type of power input : lost work or viscous loss

The viscous dissipation is the viscous losses per unit volume

dissipation in laminar pipe flow : $\frac{P}{\pi D^2 L/4} = \frac{1}{2}\eta\left(\frac{8V}{D}\right)^2$

where $\frac{8V}{D}$ is the wall shear rate

Optimal Pipe Diameter:

The economic trade-off between
the increased capital cost of large-diameter pipe and
the increased energy cost in pumping through small-diameter pipe.

The annual cost of pipe : $C_{pipe} = \Lambda D^n L$

The annual cost of power : $C_{power} \propto Q|\Delta p|$

using the Blasius eq'n,

$$C_{power} \propto L \frac{\rho^{0.75}\eta^{0.25}Q^{2.75}}{D^{4.75}}$$

$$= \lambda L \frac{\rho^{0.75}\eta^{0.25}Q^{2.75}}{D^{4.75}}$$

The total annual cost, C, is

$$C = C_{pipe} + C_{power}$$

$$= \Lambda D^n L + \lambda L \frac{\rho^{0.75}\eta^{0.25}Q^{2.75}}{D^{4.75}}$$

The minimum cost at some intermediate value of D when

$$\frac{dC}{dD} = n\Lambda D^{n-1}L - 4.75\lambda L\frac{\rho^{0.75}\eta^{0.25}Q^{2.75}}{D^{5.75}} = 0$$

or $$D = \left(\frac{4.75\lambda}{n\Lambda}\rho^{0.75}\eta^{0.25}Q^{2.75}\right)^{1/(4.75+n)}$$

$$\upsilon = \frac{4}{\pi}\left(\frac{n\Lambda}{4.75\lambda}\right)^{1/2.75}\frac{D^{(n-0.75)/2.75}}{\rho^{3/11}\eta^{1/11}} \approx 1.5 \text{ m/s}$$

(Figure 3-4. Cost per linear meter of standard steel pipe.)

3.5 Commercial Pipe

Relative Roughness:

The characteristic size of the surface roughness : k

One additional dimensionless group,

k/D (relative roughness), is required.

(Figure 3-5. Schematic of pipe with walls of uniform, regular roughness.)

Then, the dimensional analysis gives

$$f = f\left(\mathrm{Re}, \frac{k}{D} \right)$$

(Figure 3-6. Data of Nikuradse, 1950.)

- In the laminar flow, f is independent of k/D.
- Hydaulically smooth region
- Complete turbulence region : constant f
  - ⇒ the pressure drop comes entirely from the inertial forces.

Pipe roughness:

Irregularities of the surface of real pipe cannot be characterized by a single number, k/D.

Nevertheless, data for real commercial pipe are commonly analyzed in terms of k/D, and the approach works reasonably well for clean pipes.

(Figure 3-7. f vs. Re for incompressible Newtonian fluid flow
in commercial rough pipe.)

Colebrook formula (empirical)

$$\frac{1}{\sqrt{f}} = -4.0 \log_{10}\left(\frac{k}{D} + \frac{4.67}{\mathrm{Re}\sqrt{f}}\right) + 2.28$$

$$f \approx 0.04\,\mathrm{Re}^{-0.16} \qquad 4000 \le \mathrm{Re} \le 2 \times 10^7$$

Table 3-2. Typical vales of k for various types of pipe (given by Moody)

Material	k (mm)
Drawn tubing(brass, lead, glass, etc.)	1.5×10^{-3}
Commercial steel or wrought iron	0.05
Asphalted cast iron	0.12
Galvanized iron	0.15
Casr iron	0.46
Wood stave	0.2-0.9
Concrete	0.3-3
Riveted steel	0.9-9

Ex. 3.3 Water at 20℃ is pumped through a 50-mm commercial steel pipe at a velocity of 1.5 m/s. $\Delta p/L = ?$

$\eta = 10^{-3}\,\mathrm{Pa\cdot s}$, $\rho = 10^{3}\,\mathrm{kg/m^{3}}$

T h u s ,

$$Re = \frac{Dv\rho}{\eta} = \frac{(50\times10^{-3}\,\mathrm{m})(1.5\,\mathrm{m/s})(10^{3}\,\mathrm{kg/m^{3}})}{10^{-3}\,\mathrm{Pa\cdot s}} = 7.5\times10^{4}$$

From Table 3-2 we have $k = 0.05\,\mathrm{mm}$, so $k/D = 10^{-3}$.

Thus, from Fig. 3-7, $f = 0.0058$. And from the definition of f, we have $\frac{|\Delta p|}{L} = \frac{2\rho v^{2} f}{D} = 520\,\mathrm{Pa/m}$.

In a smooth pipe, $k = 0$, $f = 0.0047$ $\Rightarrow$ $\frac{\Delta p}{L} = 420\,\mathrm{Pa/m}$.

Nominal and Real Diameter:

The diameter of commercial pipe is a nominal size.

Table 3-3 Wall thickness and inside diameter of nominal 50-mm (2-in.) pipe. (O.D. = 60.33 mm)

Schedule number	Wall thickness (mm)	Inner diameter (mm)
5S	1.65	57.02
10S	2.77	54.70
40ST, 40S	3.91	52.50
80ST, 80S	5.54	49.25
160	8.74	42.85
xx	11.07	38.18

3.6 Noncircular Cross Sections

Circular pipes : volume of liquid = $\pi D^2 L/4$

surface wetted by liquid = πDL

$$\Rightarrow \quad D = \frac{4 \text{ x volume of liquid}}{\text{surface wetted by liquid}}$$

For any conduits : $$D_H = \frac{4 \text{ x volume of liquid}}{\text{surface wetted by liquid}}$$

: Hydraulic diameter

For channels of constant cross-sectional area :

$$D_H = \frac{4 \text{ x cross sectional area}}{\text{wetted perimeter}}$$

Ex. A rectangular channel with sides a and b:

$$D_H = \frac{4ab}{2(a+b)} = 2\left(\frac{1}{a} + \frac{1}{b}\right)^{-1}$$

(Figure 3-8. f vs. Re for incompressible Newtonian fluid flow in a channel with an equilateral triangular cross section.)