8. One-Dimensional Flows

8.1 Introduction

* Navier-Stokes eq'ns and continuity eq'n are four coupled, nonlinear PDE. ⇒ extremely difficult system
* In most situations of practical interest, a good deal of approximation is required, often followed by numerical computation.

The general procedure for the solution of flow problems is

1. Utilize understanding of the process to determine upon which independent variables each of the velocity components depends.
   ⇒ kinematic assumptions
2. Substitute into the continuity eq'n to ensure that the dependence assumed in step 1 is consistent with continuity.
3. Substitute into the momentum or Navier-Stokes equations and solve them.

8.2 Plane Poiseuille Flow

Problem description:

Consider the flow of an incompressible Newtonian fluid in the x direction at a steady state through a rectangular channel of very large aspect ratio.

$H/W \ll 1$: negligible side wall effects

$H/L \ll 1$: negligible entrance and exit effects $\Rightarrow$ fully developed

The flow is not turbulent.

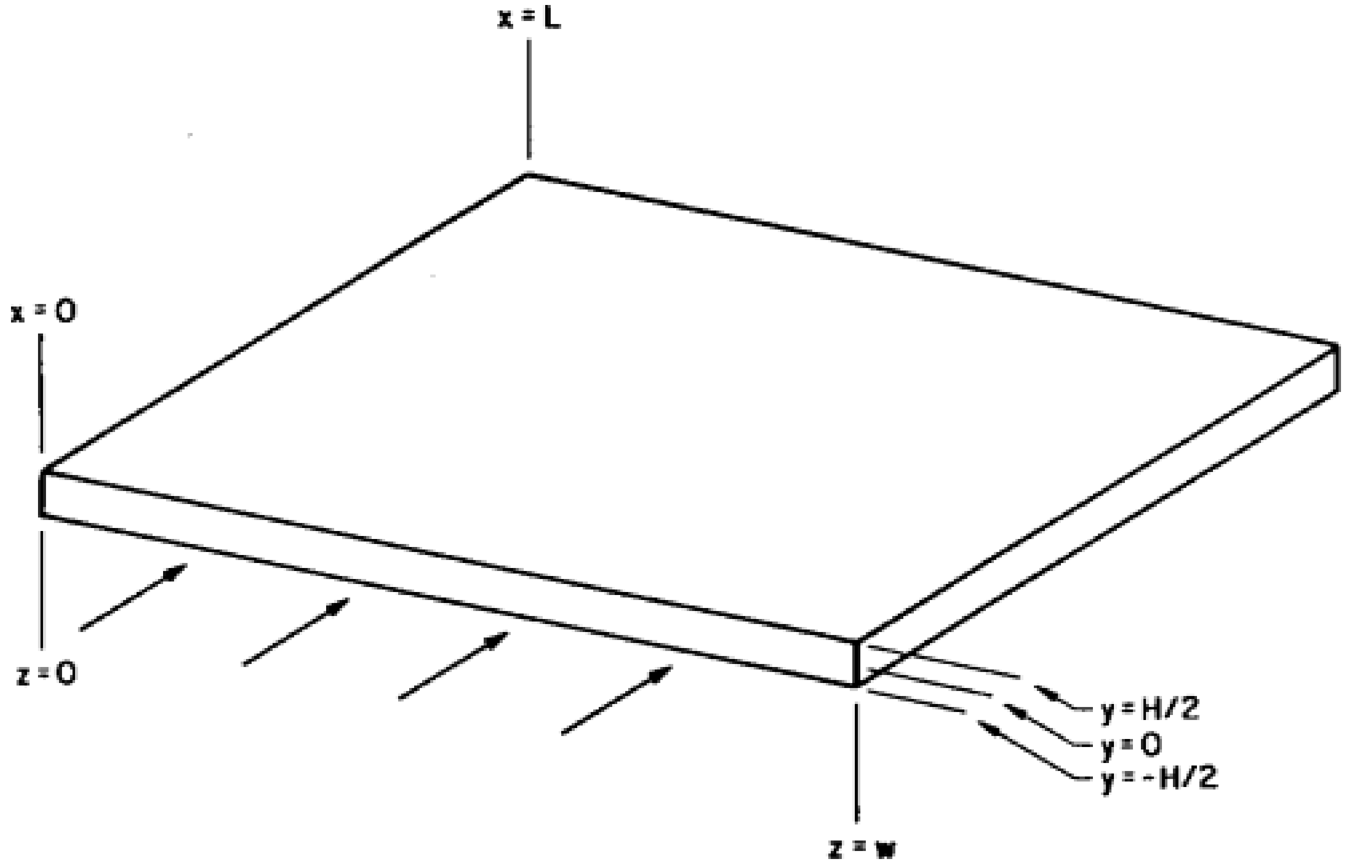

Fig. 8-1. Schematic of flow in a plane channel with large aspect ratio.

Direct solution:

The flow is fully developed. ⇒ no variations in x direction

$$\frac{\partial v_x}{\partial x} = \frac{\partial v_y}{\partial x} = \frac{\partial v_z}{\partial x} = 0$$

No side wall effects ⇒ no variations in z direction

$$\frac{\partial v_x}{\partial z} = \frac{\partial v_y}{\partial z} = \frac{\partial v_z}{\partial z} = 0$$

Therefore, $v_x = v_x(y)$, $v_y = v_y(y)$, $v_z = v_z(y)$

We further anticipate $v_y = v_z = 0$.

The continuity eq'n in Cartesian coordinates is automatically satisfied.

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0$$

Navier-Stokes eq'ns:

x component : $0 = -\frac{\partial P}{\partial x} + \eta \frac{d^2 \upsilon_x}{dy^2}$

y component : $0 = -\frac{\partial P}{\partial y}$

z component : $0 = -\frac{\partial P}{\partial z}$

Therefore, $P = P(x)$,

and $\eta \frac{d^2 \upsilon_x}{dy^2} = \frac{dP}{dx}$

function of y only = function of x only = constant = $\Delta P/L$

where $\Delta P = P_{x=L} - P_{x=0} < 0$

After the integration, we have

$$\upsilon_x = \frac{1}{2\eta} \frac{\Delta P}{L} y^2 + C_1 y + C_2$$

Apply the no-slip boundary conditions to determine C_1 and C_2.

$$\upsilon_x = 0 \quad \text{at} \quad y = \pm H/2$$

$$\Rightarrow \quad C_1 = 0 \quad \text{and} \quad C_2 = -\frac{1}{2\eta}\frac{\Delta P}{L}\frac{H^2}{4}$$

$$\therefore \ \upsilon_x = \frac{H^2}{8\eta}\left(-\frac{\Delta P}{L}\right)\left[1-\left(\frac{2y}{H}\right)^2\right]$$

: parabolic profile

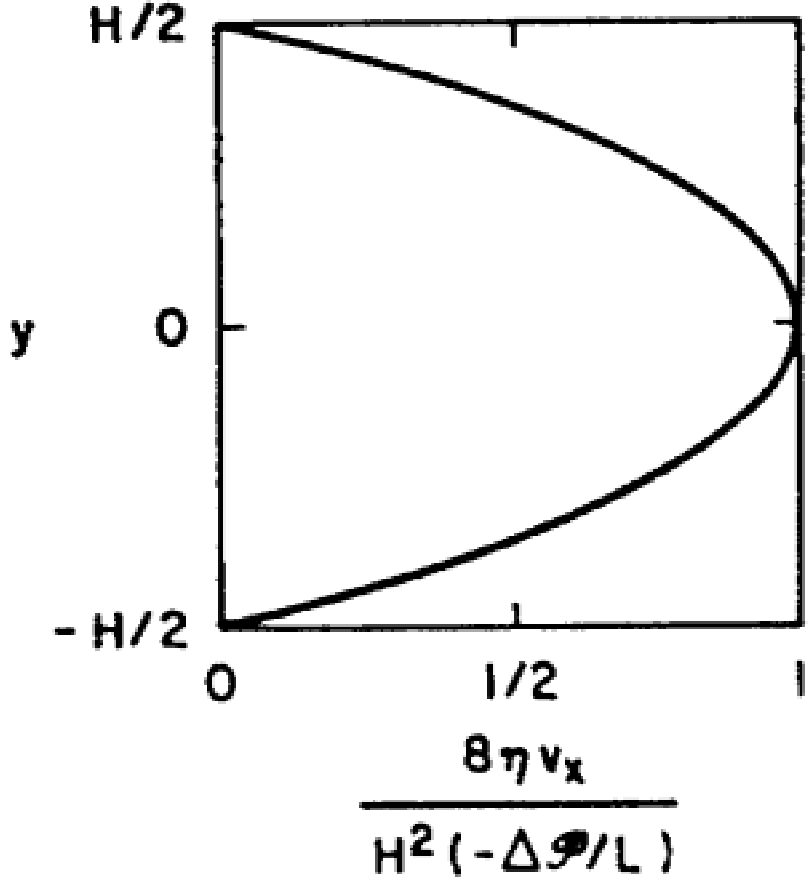

Fig. 8-2. Velocity profile for laminar flow with a pressure gradient between infinite stationary planes.

The flow rate Q is

$$Q = WH \langle \upsilon_x \rangle = \int_{area} \upsilon_x dA = W \int_{-H/2}^{H/2} \upsilon_x dy$$

or the average velocity $\langle \upsilon_x \rangle$ is

$$\langle \upsilon_x \rangle = \frac{1}{H} \int_{-H/2}^{H/2} \upsilon_x dy = \frac{H^2}{8\eta H} \left(-\frac{\Delta P}{L}\right) \int_{-H/2}^{H/2} \left[1 - \left(\frac{2y}{H}\right)^2\right] dy$$

$$= \frac{H^2}{16\eta} \left(-\frac{\Delta P}{L}\right) \int_{-1}^{1} (1 - \xi^2) d\xi = \frac{H^2}{12\eta} \left(-\frac{\Delta P}{L}\right)$$

Therefore,

$$\upsilon_x = \frac{3}{2} \langle \upsilon_x \rangle \left[1 - \left(\frac{2y}{H}\right)^2\right]$$

Symmetry boundary conditions:

$$\tau_{yx} = \frac{d\upsilon_x}{dy} = 0 \qquad \text{at} \quad y = 0$$

Relaxed assumptions:

Kinematic assumption, $\upsilon_x = \upsilon_x(y)$, $\upsilon_y = \upsilon_y(y)$, $\upsilon_z = \upsilon_z(y)$.

Then, from the continuity $\dfrac{\partial \upsilon_y}{\partial y} = 0$ or $\upsilon_y =$ constant .

Since $\upsilon_y = 0$ at $y = \pm H/2$, υ_y must be zero everywhere.

The Navier-Stokes eq′ns then reduce to

x component : $0 = -\dfrac{\partial P}{\partial x} + \eta \dfrac{d^2 \upsilon_x}{dy^2}$

y component : $0 = -\dfrac{\partial P}{\partial y}$: $P = P(x,z)$

z component : $0 = -\dfrac{\partial P}{\partial z} + \eta \dfrac{d^2 \upsilon_z}{dy^2}$

or x component : $\eta \frac{d^2 \upsilon_x}{dy^2} = \frac{\partial P}{\partial x}$

z component : $\eta \frac{d^2 \upsilon_z}{dy^2} = \frac{\partial P}{\partial z}$

Integrating the z component eq′n,

$$\upsilon_z = \frac{H^2}{8\eta}\left(-\frac{\partial P}{\partial z}\right)\left[1-\left(\frac{2y}{H}\right)^2\right]$$

and $\langle \upsilon_z \rangle = \frac{1}{H}\int_{-H/2}^{H/2} \upsilon_z dy = \frac{H^2}{12\eta}\left(-\frac{\partial P}{\partial z}\right)$

Since there is no net flow in the z direction, $\langle \upsilon_z \rangle = 0$,

and therefore, $\frac{\partial P}{\partial z} = 0$. Finally we have $\upsilon_z = 0$

Solution logic:

1. Assumptions on the nature of the velocity field.
   (kinematic assumptions)
2. Solve the continuity and Navier-Stokes eq′ns.
3. Check the initial assumptions to have internal consistency.

Set of nonlinear PDE

$\Rightarrow$ absence of the proof of uniqueness of the solution

$\Rightarrow$ the possibility of another solution

$\Rightarrow$ requires the experimental check

8.3 Plane Couette Flow (Drag Flow)

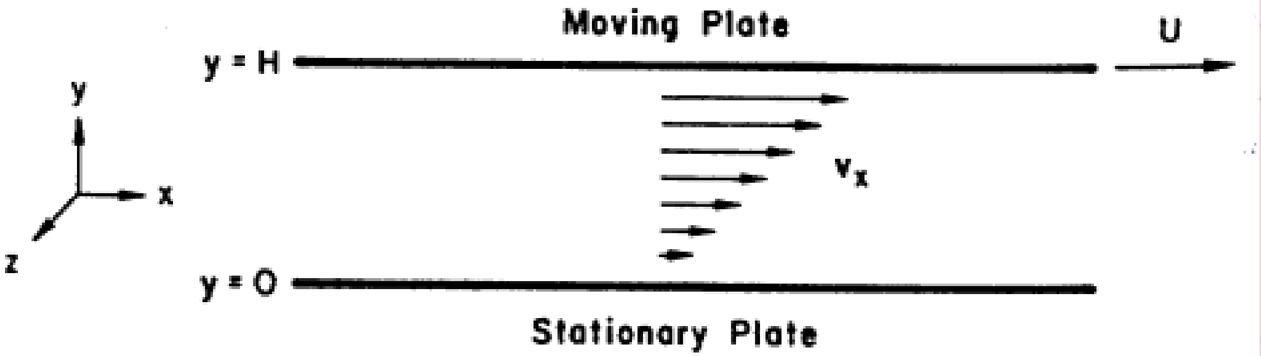

Fig. 8-3. Schematic of plane Couette flow.

Kinematic assumption:

$\upsilon_x = \upsilon_x(y)$, $\upsilon_y = \upsilon_z = 0$ which satisfies the continuity eq'n.

The Navier-Stokes eq'ns:

x component $0 = -\frac{\partial P}{\partial x} + \eta \frac{d^2 \upsilon_x}{dy^2}$

y component $0 = -\frac{\partial P}{\partial y}$, z component $0 = -\frac{\partial P}{\partial z}$

$\Rightarrow \quad P = P(x)$

Engineering Bernoulli eq'n:

no work, no losses, $\langle \upsilon_x \rangle$ is independent of x,

$\therefore$ P is also independent of x $\Rightarrow$ we assume that $\dfrac{\partial P}{\partial x} = 0$

Then, we have $\dfrac{d^2 \upsilon_x}{dy^2} = 0 \quad \Rightarrow \quad \upsilon_x = C_1 y + C_2$

Boundary conditions:

$\upsilon_x = 0$ at $y = 0 \qquad \Rightarrow \qquad C_2 = 0$

$\upsilon_x = U$ at $y = H \qquad \Rightarrow \qquad C_1 = U/H$

Therefore, $\upsilon_x = \dfrac{Uy}{H}$ and the shear stress is

$$\tau_{yx} = \eta\left(\frac{\partial \upsilon_x}{\partial y} + \frac{\partial \upsilon_y}{\partial x}\right) = \eta \frac{d\upsilon_x}{dy} = \eta \frac{U}{H}$$: independent of position

8.4 Poiseuille Flow

Consider steady fully developed flow of an incompressible Newtonian fluid in a long, smooth, round tube of radius R.

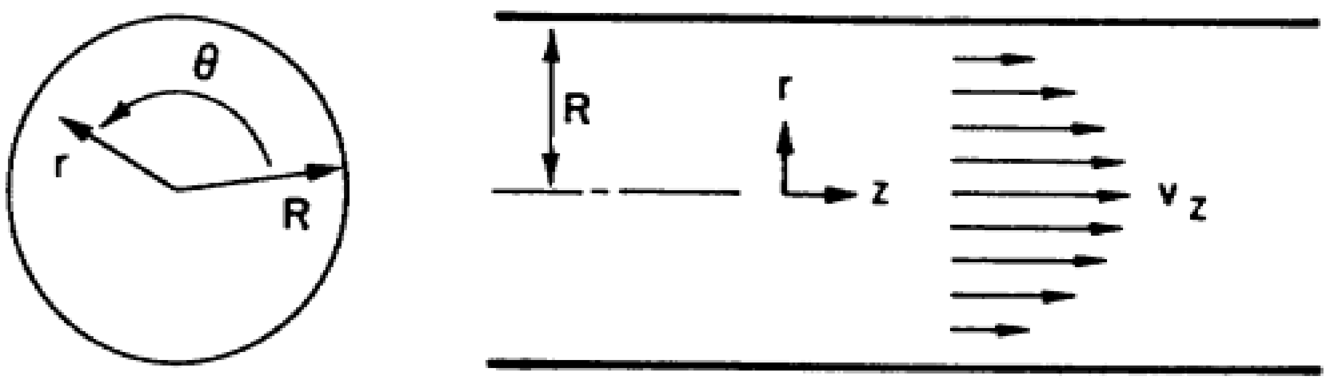

Fig. 8-4. Schematic of laminar flow in a long smooth, round tube.

Kinematic assumptions (using the cylindrical coordinates):

$$\upsilon_z = \upsilon_z(r) \quad , \qquad \upsilon_r = \upsilon_\theta = 0$$

The continuity eq'n:

$$\frac{1}{r}\frac{\partial}{\partial r}(r\upsilon_r)+\frac{1}{r}\frac{\partial \upsilon_\theta}{\partial \theta}+\frac{\partial \upsilon_z}{\partial z}=0 \quad : \text{ satisfied}$$

Navier-Stokes eq'ns:

r component: $$0=-\frac{\partial P}{\partial r}$$

θ component: $$0=-\frac{1}{r}\frac{\partial P}{\partial \theta}$$

z component: $$0=-\frac{\partial P}{\partial z}+\eta\frac{1}{r}\frac{d}{dr}\left(r\frac{d\upsilon_z}{dr}\right)$$

r and θ component eq'ns $\Rightarrow$ $P=P(z)$

and z component eq'n $\Rightarrow$ $$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\upsilon_z}{dr}\right)=\frac{1}{\eta}\frac{dP}{dz}$$

f'n of r only f'n of z only

Therefore, $\frac{1}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right) = \frac{1}{\eta}\frac{dP}{dz} = \text{constant} = \frac{1}{\eta}\frac{\Delta P}{L}$

: pressure gradient is linear in pipe length

We rewrite the z component eq'n as

$$\frac{d}{dr}\left(r\frac{dv_z}{dr}\right) = \frac{1}{\eta}\frac{\Delta P}{L}r$$

After first integration,

$$r\frac{dv_z}{dr} = \frac{1}{2\eta}\frac{\Delta P}{L}r^2 + C_1$$

or $$\frac{dv_z}{dr} = \frac{1}{2\eta}\frac{\Delta P}{L}r + \frac{C_1}{r}$$

After second integration,

$$v_z = \frac{1}{4\eta}\frac{\Delta P}{L}r^2 + C_1 \ln r + C_2$$

Since υ_z is finite at $r = 0 \quad \Rightarrow \quad C_1 = 0$

No-slip boundary condition:

$$\upsilon_z = 0 \text{ at } r = R \qquad \Rightarrow \qquad 0 = \frac{1}{4\eta}\frac{\Delta P}{L}R^2 + C_2$$

Therefore, the velocity profile becomes

$$\upsilon_z(r) = \frac{R^2}{4\eta}\left(-\frac{\Delta P}{L}\right)\left[1-\left(\frac{r}{R}\right)^2\right] \qquad = 2\langle\upsilon_z\rangle\left[1-\left(\frac{r}{R}\right)^2\right]$$

The average velocity is

$$\langle\upsilon_z\rangle = \frac{1}{\pi R^2}\int_{area}\upsilon_z dA = \frac{R^2}{8\eta}\left(-\frac{\Delta P}{L}\right)$$

: equivalent to the Hagen-Poiseuille eq'n

The friction factor:

$$f = \frac{(-\Delta P)R}{\rho\langle\upsilon_z\rangle^2 L} = \frac{8\eta}{\rho\langle\upsilon_z\rangle R} = \frac{16\eta}{\rho\langle\upsilon_z\rangle D} = \frac{16}{\text{Re}}$$

Range of solution:

The pipe must be long relative to an entry length of approximately $0.055D\,\mathrm{Re}$.

Re < 2100,   becomes unstable solution for Re > 2100. (turbulence)

Stability of steady-state solution:

$$\frac{dv}{dt} = (v-1)(v-2) \quad , \qquad v(0) = A$$

steady-state solution: $v = 1$ and $v = 2$

transient solution: $$v(t) = \frac{2(A-1)-(A-2)e^t}{A-1-(A-2)e^t}$$

For $A < 2$, $v(t) \to 1$ as $t \to \infty$.

For $A > 2$, $v(t) \to \infty$ as $t \to \ln\left(\frac{A-1}{A-2}\right)$.

(no steady-state)

8.5 Wire Coating

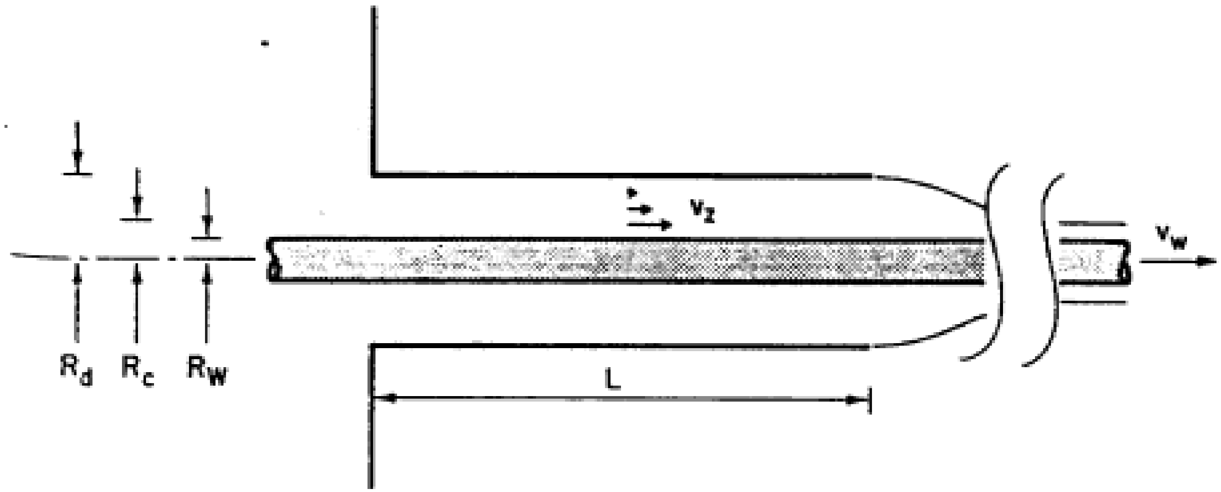

Fig. 8-5. Schematic of wire coating.

$$R_c = R_c(R_w, R_d)$$

R_c : the ultimate radius of the coated wire

R_w : the radius of the uncoated wire

R_d : the die radius

The mass balance on the coating material:

$$Q = V_w(\pi R_c^2 - \pi R_w^2) = \int_{R_w}^{R_d} 2\pi r v_z(r)\, dr$$

downstream within the die

$$\Rightarrow R_c = [R_w^2 + \frac{2}{V_w} \int_{R_w}^{R_d} r v_z(r)\, dr]^{1/2}$$

To obtain $v_z(r)$,

Kinematics: $v_z = v_z(r), \quad v_r = v_\theta = 0$

$\Rightarrow$ continuity is satisfied

Navier-Stokes eq'n:

z component $\quad 0 = \eta[\frac{1}{r}\frac{d}{dr}(r\frac{dv_z}{dr})]$

After the first integration, $r\frac{dv_z}{dr}$ = constant = C_1

The second integration : $v_z(r) = C_1 \ln r + C_2$

Apply the boundary conditions to determine C_1 and C_2

$r = R_w$: $V_w = C_1 \ln R_w + C_2$

$r = R_d$: $0 = C_1 \ln R_d + C_2$

$$\Rightarrow \quad C_1 = \frac{V_w}{\ln R_w/R_d} \quad , \quad C_2 = -\frac{V_w \ln R_d}{\ln R_w/R_d}$$

$$\therefore \quad v_z(r) = V_w \frac{\ln r/R_d}{\ln R_w/R_d}$$: independent of viscosity η

Finally we obtain

$$R_c = \left(\frac{R_d^2 - R_w^2}{2\ln R_d/R_w}\right)^{1/2}$$: independent of η and V_w

The force required to pull the wire through the die:

$$F_w = 2\pi R_w L \tau_{rz}|_{r=R_w}$$

where $\tau_{rz}|_{r=R_w} = \eta\left(\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z}\right)\Big|_{r=R_w} = \frac{\eta V_w}{R_w \ln R_w/R_d}$

$$\therefore \quad F_w = \frac{2\pi\eta V_w L}{\ln R_w/R_d} \quad < 0$$

8.6 Torsional Flow

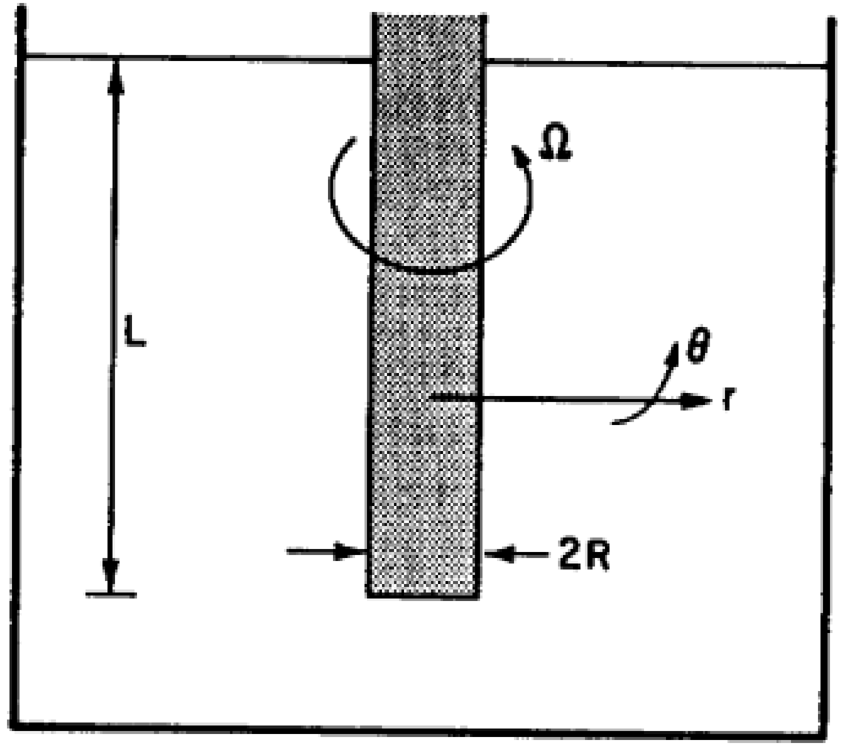

Fig. 8-6. Schematic of torsional flow.

$L \gg R$, neglect end effects

Kinematic assumption :

$v_\theta = v_\theta(r), \quad v_r = v_z = 0$ $\Rightarrow$ satisfies the continuity

Navier-Stokes eq'n :

r component $$-\rho \frac{v_\theta^2}{r} = -\frac{\partial P}{\partial r}$$

θ component $$0 = \eta \frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(r v_\theta)\right]$$

z component $$0 = -\frac{\partial P}{\partial z}$$: P is independent of z

The integration of θ component eq'n gives

$$\frac{1}{r}\frac{d}{dr}(r v_\theta) = \text{constant} = C_1 \qquad \text{or} \qquad \frac{d}{dr}(r v_\theta) = C_1 r$$

$$\Rightarrow \quad v_\theta = \frac{1}{2} C_1 r + \frac{C_2}{r}$$

: increases without bound as r increases

$\therefore \; C_1 = 0$

Apply no-slip boundary condition to determine C_2

$$r = R: \quad \upsilon_\theta = R\Omega = \frac{C_2}{R} \qquad \Rightarrow \qquad \upsilon_\theta = \frac{R^2\Omega}{r}$$

The shear stress on the cylinder wall :

$$\tau_{r\theta}|_{r=R} = \eta\left[r\frac{\partial}{\partial r}\left(\frac{\upsilon_\theta}{r}\right) + \frac{1}{r}\frac{\partial \upsilon_r}{\partial \theta}\right]\Big|_{r=R} = -2\eta\Omega$$

The total torque on the cylinder :

$$G = \int_0^{2\pi} LR^2 \tau_{r\theta}|_{r=R} d\theta = -4\pi R^2 L\eta\Omega$$

Pressure and cavitation :

r component eq'n $\Rightarrow$ $$\frac{dP}{dr} = \frac{\rho v_\theta^2}{r} = \frac{\rho (R^2\Omega)^2}{r^3}$$

After the integration, $$P = P_0 - \frac{\rho (R^2\Omega)^2}{2r^2}$$

* P_0 : maximum pressure at $r \to \infty$

* minimum pressure at the cylinder surface

$$= P_0 - \frac{1}{2}\rho (R\Omega)^2$$

The cavitation will occur if $P_0 - \frac{1}{2}\rho (R\Omega)^2 < P_{vp}$

or $$R\Omega \geq \sqrt{\frac{2(P_0 - P_{vp})}{\rho}}$$

8.8 Tube Flow of a Power-law Fluid

Power-law fluid :

In plane Couette flow : $\eta(\Gamma_s) = K|\Gamma_s|^{n-1}$

General form in three-dimensional flow : $\eta = K\left|\frac{1}{2}\,\Pi\right|^{(n-1)/2}$

Kinematic assumption :

$v_z = v_z(r), \quad v_r = v_\theta = 0$ $\Rightarrow$ satisfies the continuity

nonzero stress components : $\tau_{rz}(r)$

The momentum eq'ns :

r component: $0 = -\dfrac{\partial P}{\partial r}$

θ component: $0 = -\dfrac{1}{r}\dfrac{\partial P}{\partial \theta}$

z component: $0 = -\dfrac{\partial P}{\partial z} + \dfrac{1}{r}\dfrac{d}{dr}(r\tau_{rz})$

r and θ components $\Rightarrow$ $P = P(z)$

Then, z component: $\frac{1}{r}\frac{d}{dr}(r\tau_{rz}) = \frac{dP}{dz} = \frac{\Delta P}{L}$

After the integration, we obtain $\tau_{rz} = \frac{\Delta P}{2L}r + \frac{C_1}{r}$

Since the stress must remain finite at the centerline, $C_1 = 0$

$\therefore$ $\tau_{rz} = \frac{\Delta P}{2L}r$: a general result for fully developed pipe flow

In the pipe flow, $\frac{1}{2}\Pi = (\frac{dv_z}{dr})^2$ $\Rightarrow$ $\tau_{rz} = K\left|\frac{dv_z}{dr}\right|^{n-1}\frac{dv_z}{dr}$

Therefore, $K\left|\frac{dv_z}{dr}\right|^{n-1}\frac{dv_z}{dr} = \frac{\Delta P}{2L}r$

Since ΔP and $\frac{dv_z}{dr}$ are negative,

$$\frac{dv_z}{dr} = -(-\frac{\Delta P}{2KL})^{1/n}r^{1/n}$$

After the integration,

$$\upsilon_z(r) = \frac{n}{n+1}\left[\frac{R^{n+1}}{2K}\left(\frac{-\Delta P}{L}\right)\right]^{1/n}\left[1-\left(\frac{r}{R}\right)^{(n+1)/n}\right]$$

and $\langle\upsilon_z\rangle = \frac{n}{(3n+1)}\left[\frac{R^{n+1}}{2K}\left(-\frac{\Delta P}{L}\right)\right]^{1/n}$

υ_z can be rewritten, using $\langle\upsilon_z\rangle$, as

$$\upsilon_z(r) = \frac{3n+1}{n+1}\langle\upsilon_z\rangle\left[1-\left(\frac{r}{R}\right)^{(n+1)/n}\right]$$

The profile is blunter than that for a Newtonian fluid.

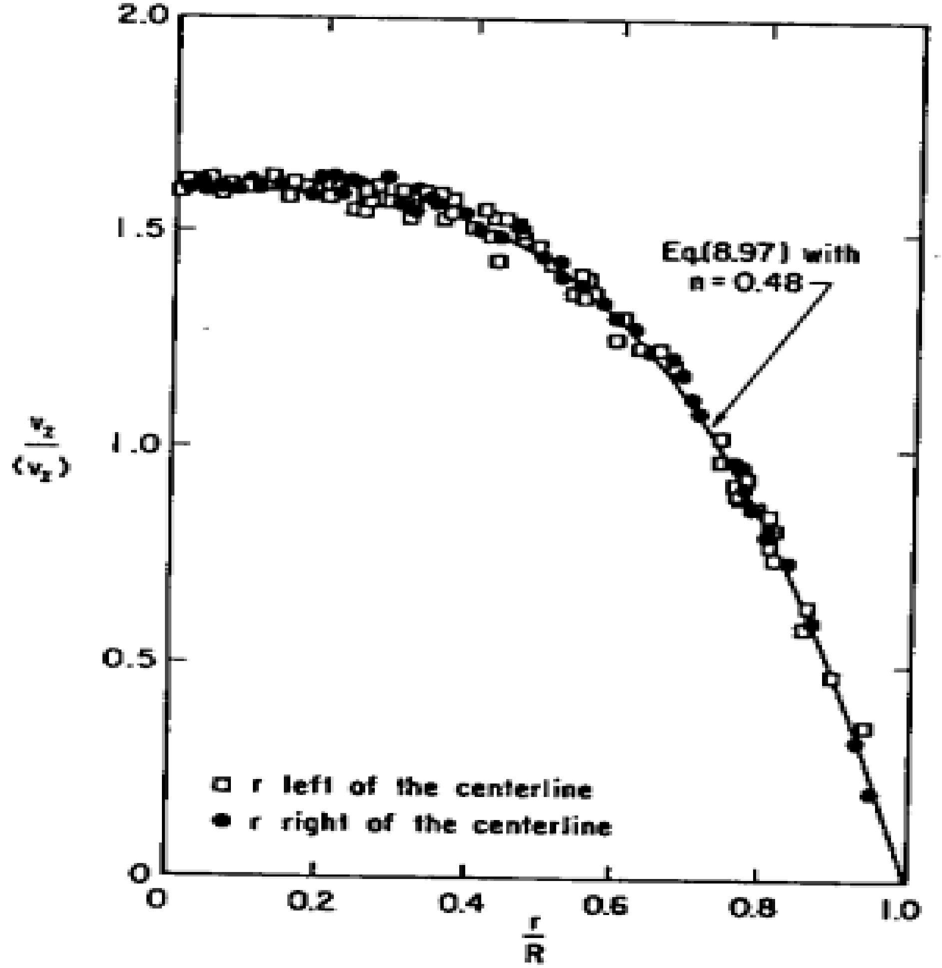

Fig. 8-8. Velocity profile in a tube for a solution of polyacrylamide in water.