9. Accelerating Flow

9.1 Introduction

Consider

* a transient startup problem for an incompressible Newtonian fluid flow
* the role of the inertial terms in the Navier-Stokes eq'ns
* the solution of a P.D.E.

We shall first state the solution without proof and examine the important physical consequences.

9.2 Problem Description

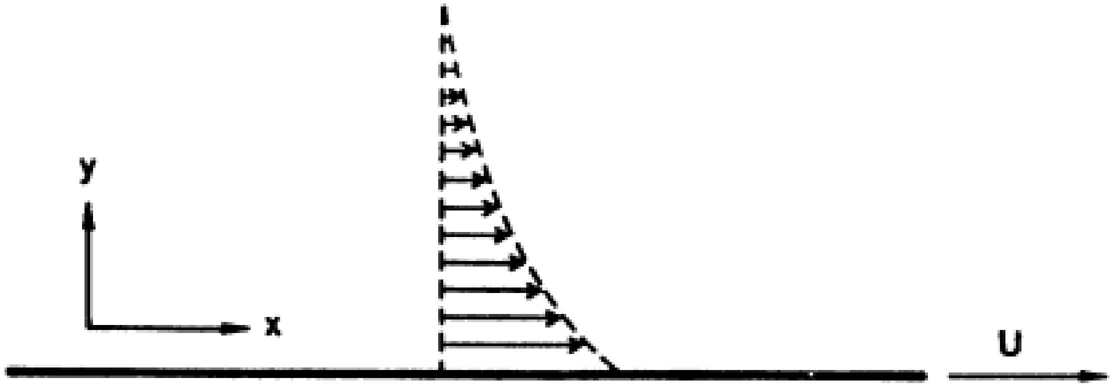

Fig. 9-1. Schematic of a plate in an infinite extent of fluid.

* For all time $t \leq 0$, the fluid and plate are at rest.
* At $t = 0^+$, the velocity of the plate in the x direction changes suddenly to a value U.

Kinematic assumption:

$$\upsilon_x = \upsilon_x(y,t), \quad \upsilon_y = \upsilon_z = 0$$: satisfies the continuity eq'n

Navier-Stokes eq'ns:

* a drag flow, no imposed pressure gradient,
no change in cross section ⇒ $\partial P/\partial x = 0$

x component $$\rho \frac{\partial \upsilon_x}{\partial t} = \eta \frac{\partial^2 \upsilon_x}{\partial y^2}$$

y component $$0 = -\frac{\partial P}{\partial y}$$

z component $$0 = -\frac{\partial P}{\partial z}$$

y and z components ⇒ P is constant

x component eq′n : * contains a nonzero inertial term
* linear in υ_x
* diffusion eq′n (cf. diffusion of mass and heat)

Initial condition: $\upsilon_x = 0$ at $t = 0 \quad y \geq 0$

Boundary conditions: $\upsilon_x = U$ at $y = 0 \quad t > 0$

$\upsilon_x = 0$ at $y = \infty \quad t < \infty$

9.3 Boundary Layer

The solution of x component eq′n :

$$\upsilon_x(y,t) = U\left[1 - \mathrm{erf}\left(\frac{y}{\sqrt{4\eta t/\rho}}\right)\right]$$

where $\mathrm{erf}(\zeta) = \frac{2}{\sqrt{\pi}} \int_0^{\zeta} e^{-\xi^2} d\xi$: the error function

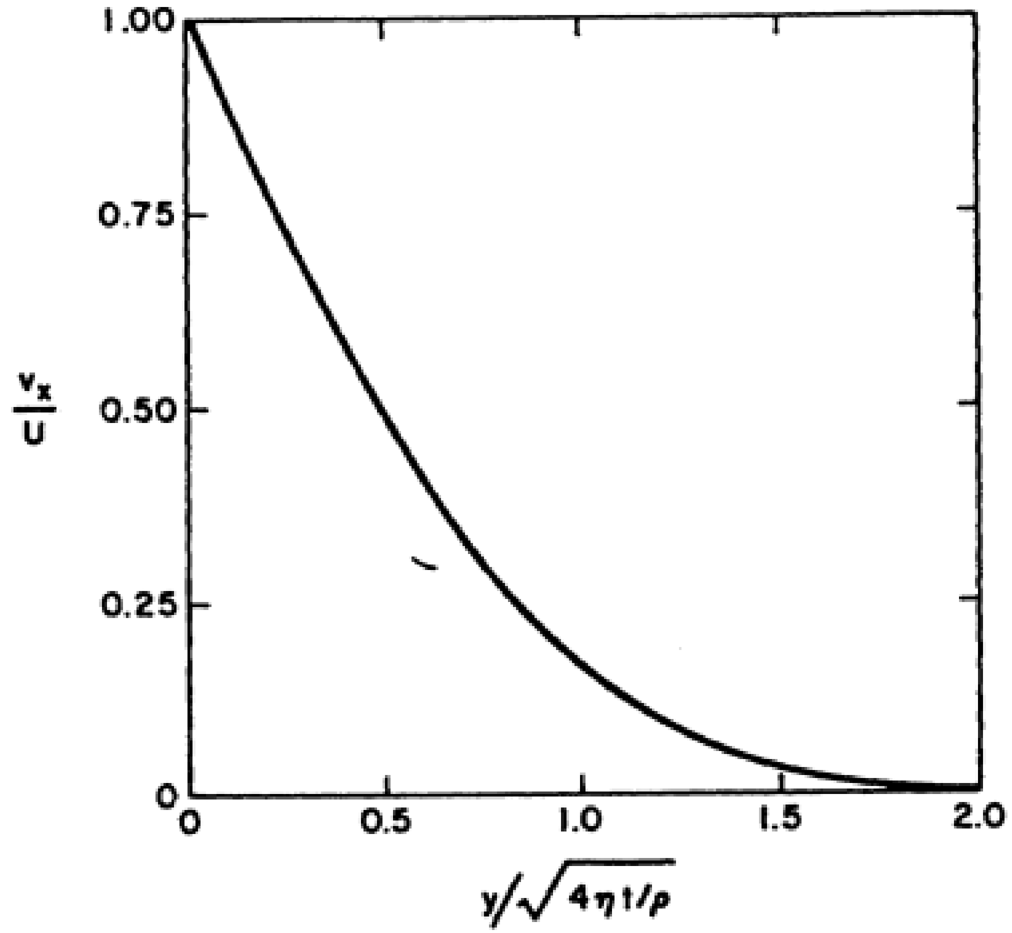

Fig. 9-2. Velocity as a function of the combined variable $y/\sqrt{4\eta t/\rho}$ for a plate suddenly set in motion.

* $\frac{\upsilon_x}{U}$ vs. $\frac{y}{\sqrt{4\eta t/\rho}}$
* $\upsilon_x \approx 0$ for $y \approx 2\sqrt{4\eta t/\rho}$

At any time the influence of the wall is transmitted by viscous shearing only a finite distance into the fluid.

The distance is proportional to $\sqrt{\frac{\eta}{\rho} t}$.

The wall region, where viscous forces are important, is known as the boundary layer.

9.4 Dimensional Analysis (Similarity) Solution

Six variables (υ_x, U, t, y, ρ, η) and three dimensions.

Therefore, there can be at most three independent dimensionless groups. $\Rightarrow \dfrac{\upsilon_x}{U}, \quad \dfrac{y^2\rho}{\eta t}, \quad \dfrac{yU\rho}{\eta} \quad \Rightarrow \quad \dfrac{\upsilon_x}{U} = f\left(\dfrac{y^2\rho}{\eta t}, \dfrac{yU\rho}{\eta}\right)$

Let's introduce a dimensionless velocity $u = \upsilon_x/U$. The x eq'n then becomes

$$\rho\frac{\partial u}{\partial t} = \eta\frac{\partial^2 u}{\partial y^2}$$

and the boundary conditions become

$$u = 0 \text{ at } t = 0 \quad y \geq 0$$

$$u = 1 \text{ at } y = 0 \quad t > 0$$

$$u = 0 \text{ at } y = \infty \quad t < \infty$$

$\Rightarrow$ u (or υ_x/U) is not a function of U.

$$\therefore \quad \frac{v_x}{U} = f\left(\frac{y^2\rho}{\eta t}\right) \quad \text{or} \quad \frac{v_x}{U} = f\left(\frac{y}{\sqrt{\eta t/\rho}}\right)$$

relative velocity — relative distance

A velocity profile, in which the relative velocity is a function only of a relative distance, is called a similar profile, or a similarity solution.

Let's define a new variable $\zeta = \dfrac{y}{\sqrt{4\eta t/\rho}}$: similarity variable

Then,

$$\frac{\partial v_x}{\partial t} = \frac{dv_x}{d\zeta}\frac{\partial \zeta}{\partial t} = -\frac{y}{\sqrt{16\eta t^3/\rho}}\frac{dv_x}{d\zeta}$$

$$\frac{\partial v_x}{\partial y} = \frac{dv_x}{d\zeta}\frac{\partial \zeta}{\partial y} = \frac{1}{\sqrt{4\eta t/\rho}}\frac{dv_x}{d\zeta}$$

$$\frac{\partial^2 v_x}{\partial y^2} = \frac{d}{d\zeta}\left(\frac{\partial v_x}{\partial y}\right)\frac{\partial \zeta}{\partial y} = \frac{1}{4\eta t/\rho}\frac{d^2 v_x}{d\zeta^2}$$

Substituting these relations into the x eq′n, we obtain the ODE:

$$-2\zeta \frac{d\upsilon_x}{d\zeta} = \frac{d^2\upsilon_x}{d\zeta^2}$$

The initial and boundary conditions:

$$\upsilon_x = U \text{ at } \zeta = 0$$

$$\upsilon_x = 0 \text{ at } \zeta = \infty$$

Solving the above ODE,

$$\frac{d}{d\zeta}\left(\ln \frac{d\upsilon_x}{d\zeta}\right) = -2\zeta \qquad \text{or} \qquad \frac{d\upsilon_x}{d\zeta} = C_1 e^{-\zeta^2}$$

The second integration then gives

$$\upsilon_x = C_1 \int_0^{\zeta} e^{-\zeta^2} d\zeta + C_2$$

After applying the B.C. to determine C_1 and C_2,

$$\upsilon_x(y,t) = U\left[1 - \mathrm{erf}\left(\frac{y}{\sqrt{4\eta t/\rho}}\right)\right]$$