

CHE302 LECTURE X
STABILITY OF CLOSED-LOOP
CONTROL SYSTEMS

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DEFINITION OF STABILITY

- **BIBO Stability**

- “An unconstrained linear system is said to be **stable** if the output response is bounded for all bounded inputs. Otherwise it is said to be **unstable**.”

- **General Stability**

- A linear system is **stable** if and only if all roots (poles) of the denominator in the transfer function are negative or have negative real parts (OLHP). Otherwise, the system is **unstable**.

⇒ What is the difference between the two definitions?

- Open-loop stable/unstable
- Closed-loop stable/unstable
- **Characteristic equation:** $1 + G_{OL}(s) = 0$
- Nonlinear system stability: Lyapunov and Popov stability

- **Supplements for stability**

- For input-output model,

- **Asymptotic stability (AS):** For a system with zero equilibrium point, if $u(t)=0$ for all time t implies $y(t)$ goes to zero with time.
 - Same as “General stability”: all poles have to be in OLHP.
 - **Marginally stability (MS):** For a system with zero equilibrium point, if $u(t)=0$ for all time t implies $y(t)$ is bounded for all time.
 - Same as BIBO stability: all poles have to be in OLHP or on the imaginary axis with any poles occurring on the imaginary axis non-repeated.
 - If the imaginary pole is repeated the mode is $t\sin(\omega t)$ and it is unstable.

- For state-space model,

- Even though there are unstable poles and if they are cancelled by the zeros exactly (**pole-zero cancellation**), the system is **BIBO stable**.
 - **Internally AS:** if $u(t)=0$ for all time t implies that $x(t)$ goes to zero with time for all initial conditions $x(0)$.
 - Cancelled poles have to be in OLHP.

EXAMPLES

- Feedback control system**

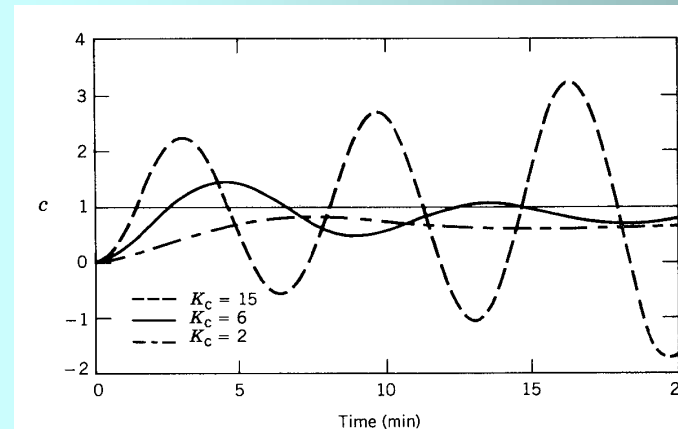
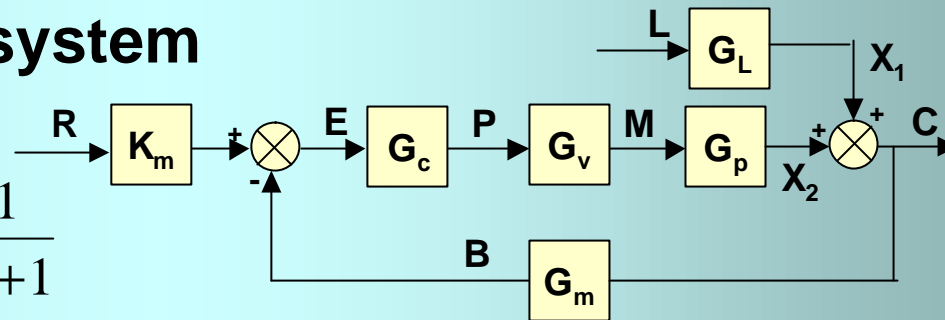
$$G_c(s) = K_c$$

$$G_v(s) = \frac{1}{2s+1} \quad G_m(s) = \frac{1}{s+1}$$

$$G_p(s) = G_L(s) = \frac{1}{5s+1}$$

$$\frac{C(s)}{R(s)} = \frac{K_m G_c G_v G_p}{1 + G_c G_v G_p G_m}$$

$$= \frac{K_c (s+1)}{10s^3 + 17s^2 + 8s + 1 + K_c}$$



- Using root-finding techniques, the poles can be calculated.
- As K_c increases, the step response gets more oscillatory.
- If $K_c > 12.6$, the step response is unstable.

- **Simple Example 1**

$$G_c(s) = K_c, \quad G_v(s) = K_v, \quad G_m(s) = 1, \quad G_p(s) = K_p / (t_p s + 1)$$

Characteristic equation: $1 + G_{OL}(s) = 1 + K_c K_v K_p / (t_p s + 1) = 0$

$$t_p s + (1 + K_c K_v K_p) = 0 \Rightarrow s = -(1 + K_c K_v K_p) / t_p$$

$$\therefore K_c K_v K_p > -1 \text{ for stability}$$

- When $K_p > 0$ and $K_v > 0$, the controller should be reverse acting ($K_c > 0$) for stability.

- **Simple example 2**

$$G_c(s) = K_c, \quad G_v(s) = 1 / (2s + 1), \quad G_m(s) = 1, \quad G_p(s) = 1 / (5s + 1)$$

Characteristic equation: $1 + K_c / [(2s + 1)(5s + 1)] = 0$

$$10s^2 + 7s + 1 + K_c = 0 \Rightarrow s = \left[-7 \pm \sqrt{49 - 40(1 + K_c)} \right] / 20$$

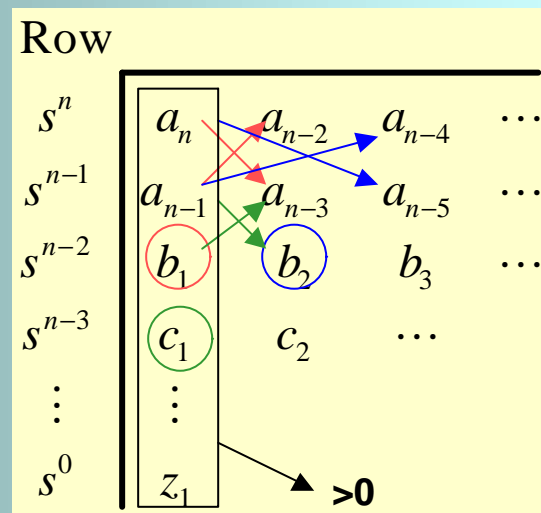
$$\therefore K_c > -1 \text{ for stability}$$

ROUTH-HURWITZ STABILITY CRITERION

- From the characteristic equation of the form:

$$a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0 = 0 \quad (a_n > 0)$$

- Construct the Routh array



$$b_1 = (a_{n-1} a_{n-2} - a_n a_{n-3}) / a_{n-1}$$

$$b_2 = (a_{n-1} a_{n-4} - a_n a_{n-5}) / a_{n-1}$$

⋮

$$c_1 = (b_1 a_{n-3} - a_{n-1} b_2) / b_1$$

$$c_2 = (b_1 a_{n-5} - a_{n-1} b_3) / b_1$$

⋮

$$b_1 = - \begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix} / a_{n-1}$$

$$b_2 = - \begin{vmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{vmatrix} / a_{n-1}$$

$$c_1 = - \begin{vmatrix} a_{n-1} & a_{n-3} \\ b_1 & b_2 \end{vmatrix} / b_1$$

$$c_2 = - \begin{vmatrix} a_{n-1} & a_{n-5} \\ b_1 & b_3 \end{vmatrix} / b_1$$

- A **necessary condition** for stability: all a_i 's are positive
- “A **necessary and sufficient condition** for all roots of the characteristic equation to have negative real parts is that all of the elements in the left column of the Routh array are positive.”

- **Example for Routh test**

- **Characteristic equation**

$$10s^3 + 17s^2 + 8s + 1 + K_c = 0$$

- **Necessary condition**

$$1 + K_c > 0 \Rightarrow K_c > -1$$

- **If any coefficient is not positive, stop and conclude the system is unstable. (at least one RHP pole, possibly more)**

- **Routh array**

s^3	10	8
s^2	17	$1 + K_c$
s^1	b_1	b_2
s^0	c_1	

$$b_1 = \frac{17(8) - 10(1 + K_c)}{17} = 7.41 - 0.588K_c$$

$$b_2 = \frac{17(0) - 10(0)}{17} = 0$$

$$c_1 = \frac{b_1(1 + K_c) - 17(0)}{b_1} = 1 + K_c$$

- **Stable region**

$$b_1 = 7.41 - 0.588K_c > 0 \text{ and } K_c > -1 \Rightarrow -1 < K_c < 12.6$$

- **Supplements for Routh test**

- It is valid only when the characteristic equation is a polynomial of s . (Time delay cannot be handled directly.)
- If the characteristic equation contains time delay, use Pade approximation to make it as a polynomial of s .
- Routh test can be used to test if the real part of all roots of characteristic equation are less than $-c$.

- **Original characteristic equation**

$$a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0 = 0 \quad (a_n > 0)$$

- **Modify characteristic equation and apply Routh criterion**

$$\begin{aligned} a_n (s + c)^n + a_{n-1} (s + c)^{n-1} + \dots + a_1 (s + c) + a_0 \\ = a'_n s^n + a'_{n-1} s^{n-1} + \dots + a'_1 s + a'_0 = 0 \end{aligned}$$

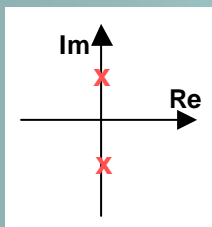
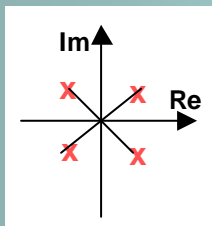
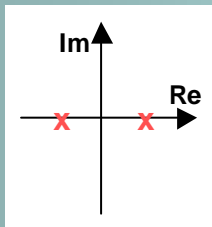
- The number of sign change in the 1st column of the Routh array indicates the number of poles in RHP.
- If the two rows are proportional or the any of 1st column is zero, Routh array cannot be proceeded.

- **Remedy for special cases of Routh array**

- **Only the pivot element is zero and others are not all zero**

- Replace zero with **positive** small number (ϵ), and proceed.
- If there is no sign change in the 1st column, it indicates there is a pair of pure imaginary roots (marginally stable). If not, the sign change indicates the no. of RHP poles.

- **Entire row becomes zero (two rows are proportional)**



- It implies the characteristic polynomial is divided exactly by the polynomial one row above (**always even-ordered polynomial**).
- Replace the row with the coefficients of the derivative (**auxiliary polynomial**) of the polynomial one row above and proceed.
- This situation indicates **at least** either **a pair of real roots symmetric about the origin** (one unstable), and/or **two complex pairs symmetric about the origin** (one unstable pair).
- **If there is no sign change after the auxiliary polynomial**, it indicated that the polynomial prior to the auxiliary polynomial has all pure imaginary roots.

DIRECT SUBSTITUTION METHOD

- Find the value of variable that locates the closed-loop poles at the imaginary axis (stability limit).

- **Example**

- Characteristic equation: $1 + G_c G_p = 10s^3 + 17s^2 + 8s + 1 + K_c = 0$

- On the imaginary axis s becomes $j\omega$.

$$-10j\omega^3 - 17\omega^2 + 8j\omega + 1 + K_{cm} = (1 + K_{cm} - 17\omega^2) + j\omega(8 - 10\omega^2) = 0$$

$$\therefore (1 + K_{cm} - 17\omega^2) = 0 \quad \text{and} \quad \omega(8 - 10\omega^2) = 0$$

$$\therefore \omega = 0 \text{ or } \omega^2 = 0.8 \Rightarrow K_{cm} = -1 \text{ or } K_{cm} = 12.6$$

- Try a test point such as $K_c = 0$

$$10s^3 + 17s^2 + 8s + 1 = (s + 1)(2s + 1)(5s + 1) = 0 \quad (\text{All stable})$$

\therefore Stable range is $-1 < K_c < 12.6$

ROOT LOCUS DIAGRAMS

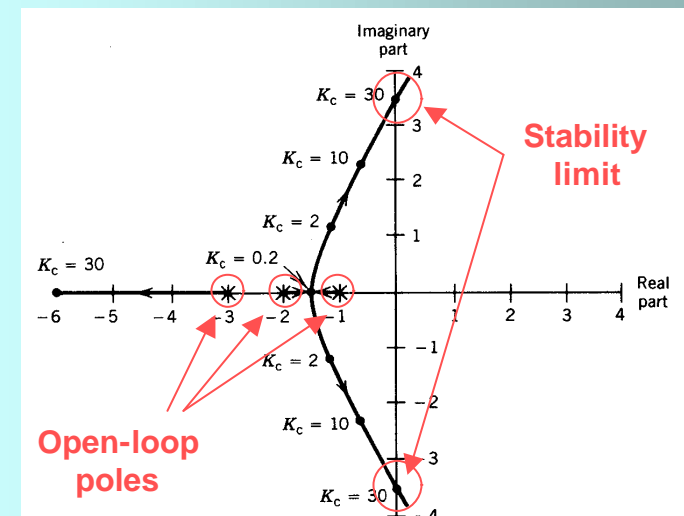
- Diagram shows the location of closed-loop poles (roots of characteristic equation) depending on the parameter value. (Single parametric study)
- Find the roots as a function of parameter
- Each loci starts at open-loop poles and approaches to zeros or $\pm\infty$.

For $G(s) = N(s) / D(s)$

$$\lim_{K_c \rightarrow 0} (D(s) + K_c N(s)) = D(s) = 0 \text{ (poles)}$$

$$\lim_{K_c \rightarrow \infty} (D(s) + K_c N(s)) = N(s) = 0 \text{ (zeros)}$$

$$\begin{aligned} \text{Ex) } (s+1)(s+2)(s+3) + 2K_c &= 0 \\ \Rightarrow K_c &< 30 \end{aligned}$$



BODE STABILITY CRITERION

- **Bode stability criterion**

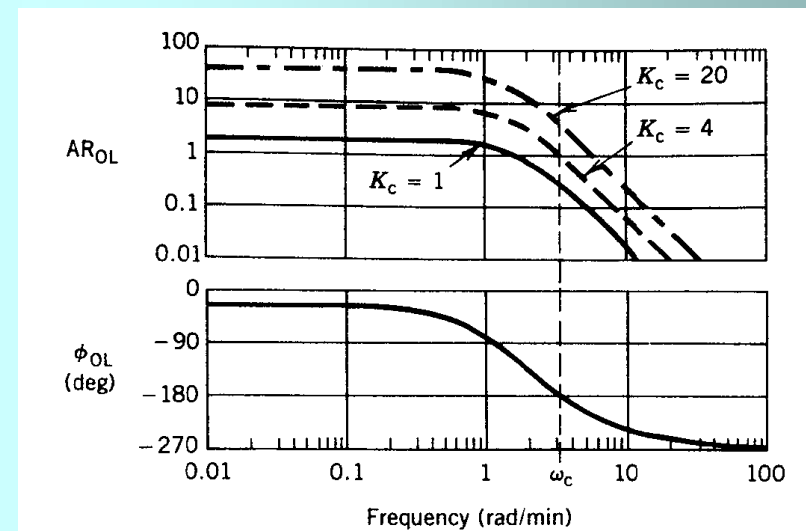
“A closed-loop system is **unstable** if the frequency response of the open-loop transfer function $G_{OL} = G_c G_v G_p G_m$ has an amplitude ratio greater than one at the critical frequency. Otherwise, the closed-loop system is stable.”

- Applicable to open-loop stable systems with only one critical frequency

- **Example:**

$$G_{OL} = \frac{2K_c}{(0.5s + 1)^3}$$

K_c	AR_{OL}	Classification
1	0.25	stable
4	1	Marginally stable
20	5	unstable



GAIN MARGIN (GM) AND PHASE MARGIN (PM)

- **Margin: How close is a system to stability limit?**

- **Gain Margin (GM)**

$$GM = 1 / AR(\omega_c)$$

- For stability, $GM > 1$

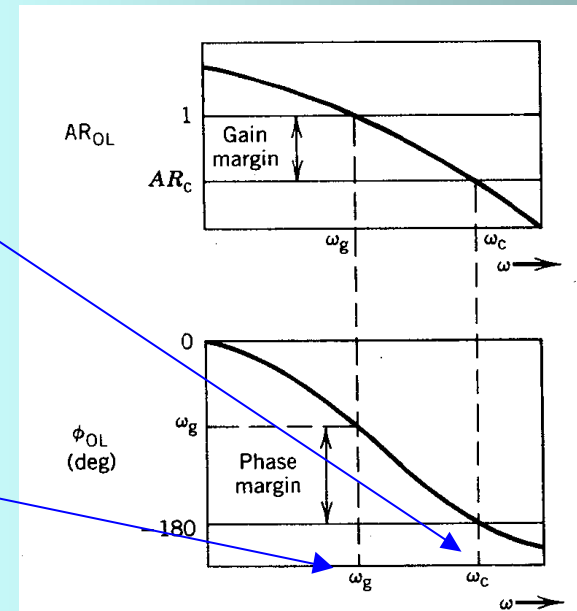
- **Phase Margin (PM)**

$$PM = f(\omega_g) + 180^\circ$$

- For stability, $PM > 0$

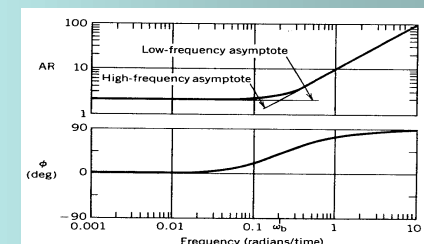
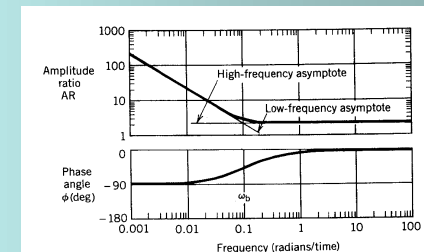
- **Rule of thumb**

- Well-tuned system: $GM=1.7-2.0$, $PM=30^\circ -45^\circ$
- Large GM and PM: sluggish
- Small GM and PM: oscillatory
- If the uncertainty on process is small, tighter tuning is possible.



EFFECT OF PID CONTROLLERS ON FREQUENCY RESPONSE

- **P**
 - As K_c increases, AR_{OL} increases (**faster** but **destabilizing**)
 - No change in phase angle
- **PI**
 - Increase AR_{OL} more at low freq.
 - As t_I decreases, AR_{OL} increases (**destabilizing**)
 - More phase lag for lower freq. (moves critical freq. toward lower freq. => usually **destabilizing**)
- **PD**
 - Increase AR_{OL} more at high freq.
 - As t_D increases, AR_{OL} increases at high freq. (**faster**)
 - More phase lead for high freq. (moves critical freq. toward higher freq. => usually **stabilizing**)



NYQUIST STABILITY CRITERION

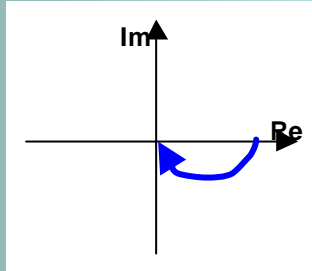
- **Nyquist stability criterion**

“If N is the number of times that the Nyquist plot encircles the point $(-1,0)$ in the complex plane in the clockwise direction, and P is the number of open-loop poles of $G_{OL}(s)$ that lies in RHP, then $Z=N+P$ is the number of unstable roots of the closed-loop characteristic equation.”

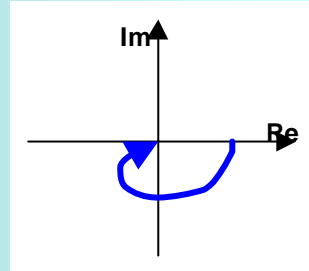
- **Applicable to even unstable systems and the systems with multiple critical frequencies**
- **The point $(-1,0)$ corresponds to $AR=1$ and $PA=-180^\circ$.**
- **Negative N indicates the encirclement of $(-1,0)$ in counterclockwise direction.**

- Some examples

1st order lag

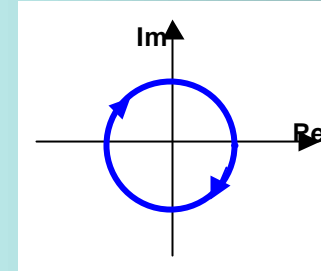


2nd order lag

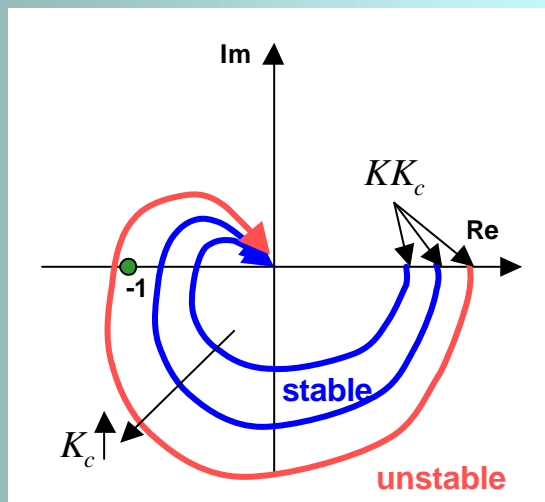


$$G(s) = \frac{K_c(2s+1)(s+1)}{s(20s+1)(10s+1)(0.5s+1)}$$

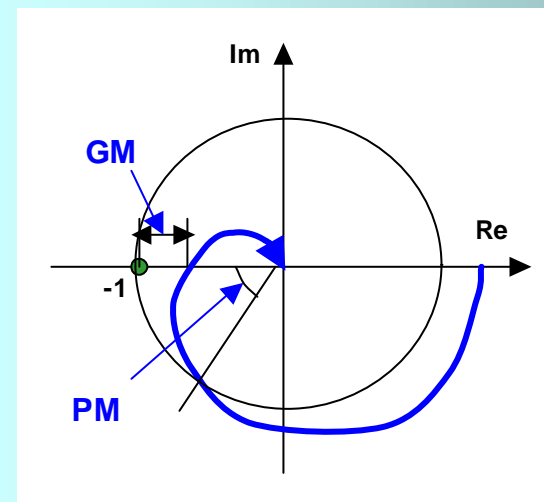
Pure time delay



Stable 3rd order lag + P control



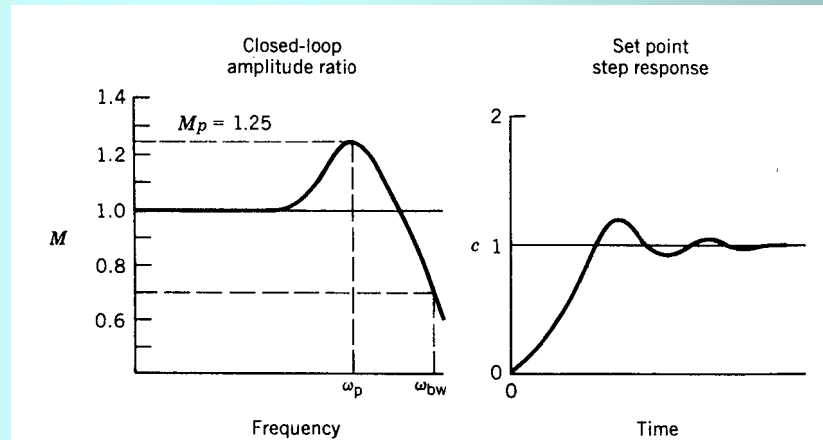
GM and PM



CLOSED-LOOP FREQUENCY RESPONSE

- Closed-loop amplitude ratio and phase angle

$$M = \left| \frac{Y(j\omega)}{R(j\omega)} \right|$$
$$y = \angle \frac{Y(j\omega)}{R(j\omega)}$$



For set point change,

- M should be unity as $\omega \rightarrow 0$. (No offset)
- M should maintain at unity up to as high a freq. as possible. (rapid approach to a new set point)
- A resonant peak (M_p) in M should be present but not greater than 1.25. (large ω_p implies faster response to a new set point)
- Large bandwidth (ω_{bw}) indicates a relatively fast response with a short rise time.

ROBUSTNESS

- **Definition**

“Despite the **small change** in the process or **some inaccuracies** in the process model, if the control system is insensitive to the uncertainties in the system and functions properly.”

- The robust control system should be, despite the certain size of uncertainty of the model,
 - Stable
 - Maintaining reasonable performance
- Uncertainty (confidence level of the model):
 - Process gain, Time constants, Model order, etc.
 - Input, output
- If uncertainty is high, the performance specification cannot be too tight: might cause even instability