# CHE302 LECTURE IX
# FREQUENCY RESPONSES

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**Fall 2001**

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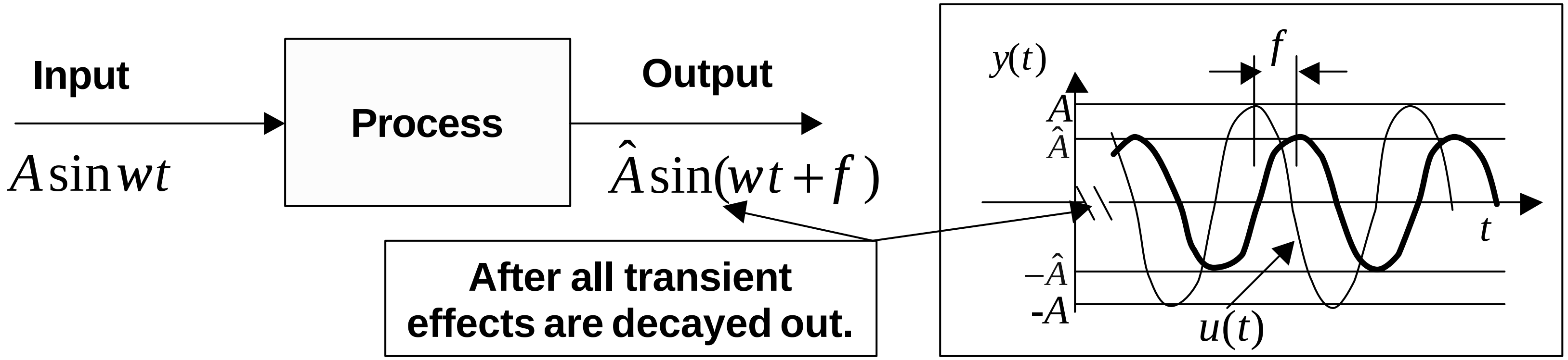
# DEFINITION OF FREQUENCY RESPONSE

- **For linear system**
  - **"The ultimate output response of a process for a sinusoidal input at a frequency will show amplitude change and phase shift at the same frequency depending on the process characteristics."**

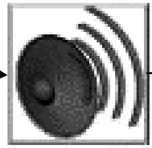



  - **Amplitude ratio (AR): attenuation of amplitude,** $\hat{A}/A$
  - **Phase angle (** $f$ **): phase shift compared to input**
  - **These two quantities are the function of frequency.**

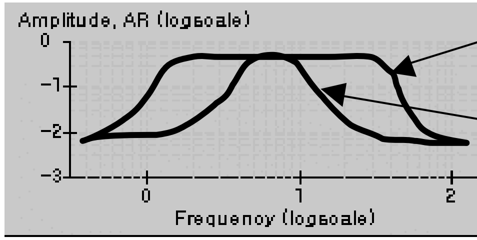
# BENEFITS OF FREQUENCY RESPONSE

- **Frequency responses are the informative representations of dynamic systems**
  - Audio Speaker

Elec. Signal



Sound wave

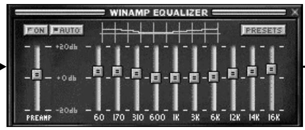



**Expensive speaker**

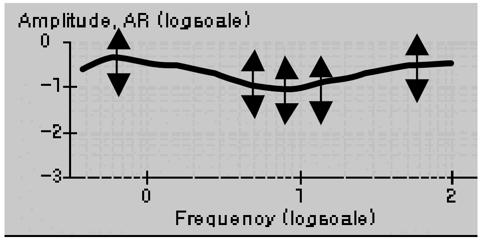
**Cheap speaker**

  - Equalizer

Raw signal

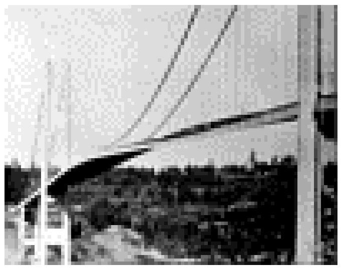



Processed signal



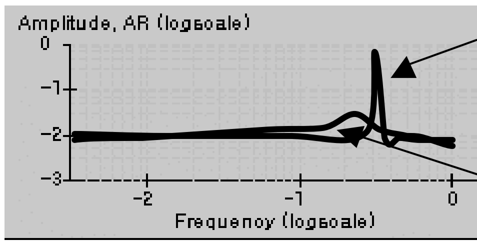

**Adjustable for each frequency band**

  - Structure



Old Tacoma bridge

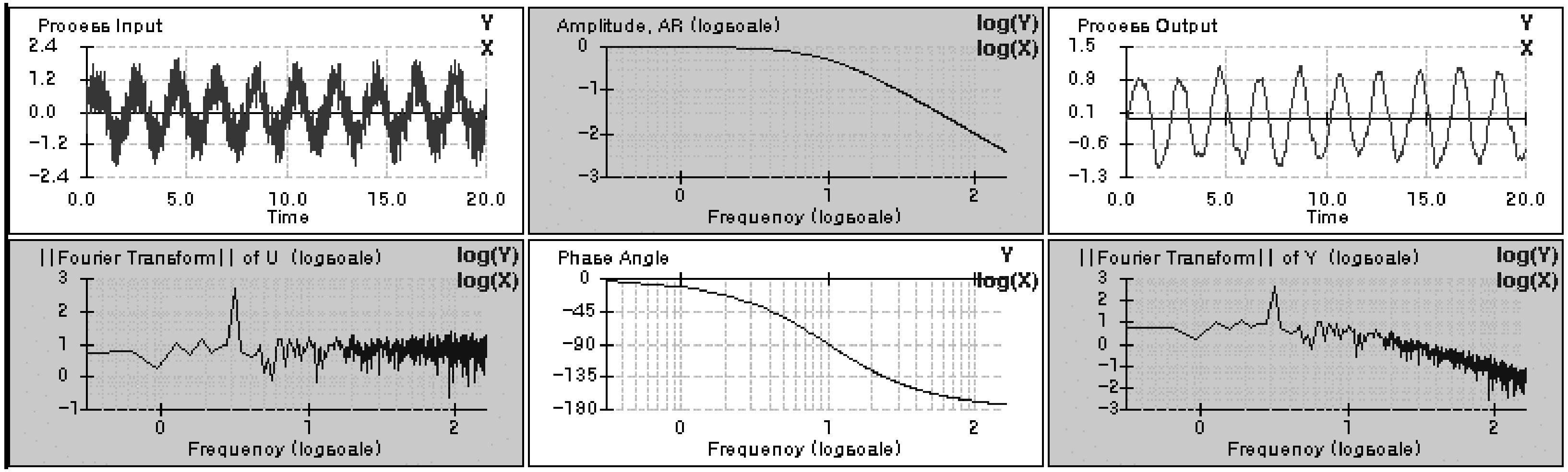
New Tacoma bridge



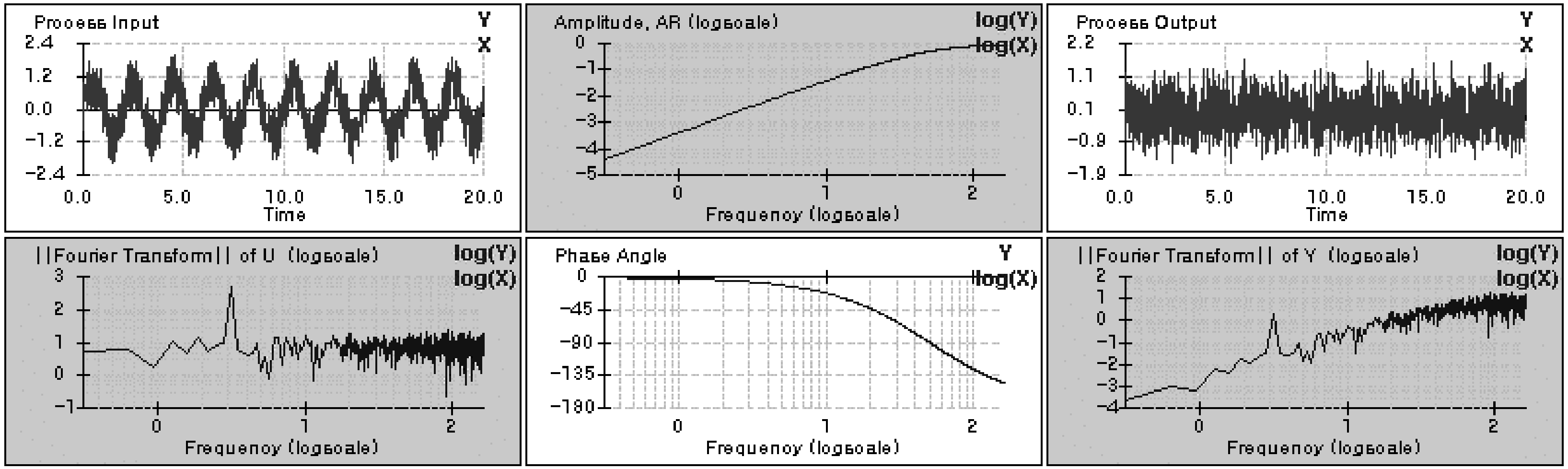

**Old Tacoma bridge**

**New Tacoma bridge**

– **Low-pass filter**




– **High-pass filter**

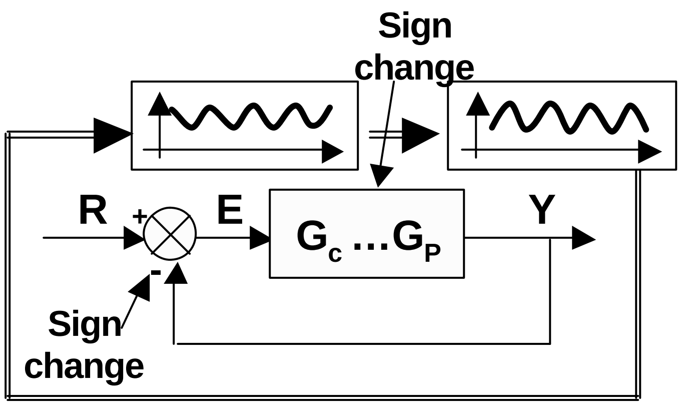



– **In signal processing field, transfer functions are called “filters”.**

- **Any linear dynamical system is completely defined by its frequency response.**
  - **The AR and phase angle define the system completely.**
  - **Bode diagram**
    - **AR in log-log plot**
    - **Phase angle in log-linear plot**
  - **Via efficient numerical technique (fast Fourier transform, FFT), the output can be calculated for any type of input.**

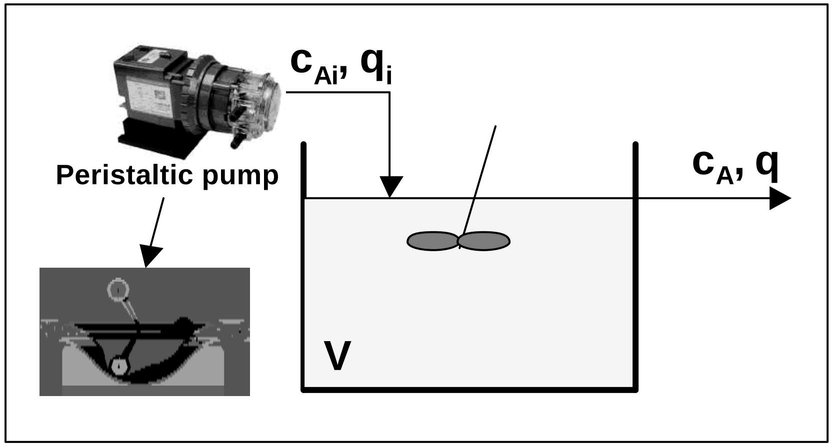
- **Frequency response representation of a system dynamics is very convenient for designing a feedback controller and analyzing a closed-loop system.**
  - **Bode stability**
  - **Gain margin (GM) and phase margin (PM)**

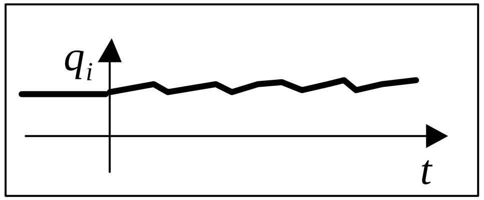
- **Critical frequency**
  - **As controller gain changes, the amplitude ratio (AR) and the phase angle (PA) change.**
  - **The frequency where the PA reaches –180° is called critical frequency ($w_c$).**
  - **The component of output at the critical frequency will have the exactly same phase as the signal goes through the loop due to comparator (-180 °) and phase shift of the process (-180 °).**
  - **For the open-loop gain at the critical frequency,** $K_{OL}(w_c)=1$
    - **No change in magnitude**
    - **Continuous cycling**
  - **For** $K_{OL}(w_c)>1$
    - **Getting bigger in magnitude**
    - **Unstable**
  - **For** $K_{OL}(w_c)<1$
    - **Getting smaller in magnitude**
    - **Stable**



# • Example

– **If a feed is pumped by a peristaltic pump to a CSTR, will the fluctuation of the feed flow appear in the output?**




$$V\frac{dc_A}{dt} = q_i c_{Ai} - q c_A \quad (q \approx \text{constant})$$

$$\frac{C_A(s)}{q_i(s)} = \frac{C_{Ai}}{Vs+q} = \frac{C_{Ai}/q}{(V/q)s+1}$$

– **$V$=50cm³, $q$=90cm³/min (so is the average of $q_i$)**

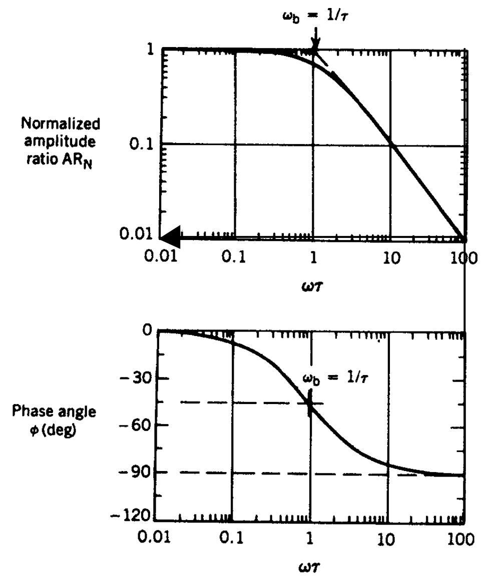
  • **Process time constant=0.555min.**

– **The rpm of the peristaltic pump is 60rpm.**

  • **Input frequency=180rad/min (3blades)**

– **The AR=0.01** $(wt = 100)$

**If the magnitude of fluctuation of $q_i$ is 5% of nominal flow rate, the fluctuation in the output concentration will be about 0.05% which is almost unnoticeable.**



# OBTAINING FREQUENCY RESPONSE

- **From the transfer function, replace $s$ with $jw$**

$$G(s) \xrightarrow{s=jw} G(jw)$$

**Transfer function** **Frequency response**

  - **For a pole, $s = a + jw$, the response mode is $e^{(a+jw)t}$.**
  - **If the modes are not unstable ($a \le 0$) and enough time elapses, the survived modes becomes $e^{jwt}$. (ultimate response)**

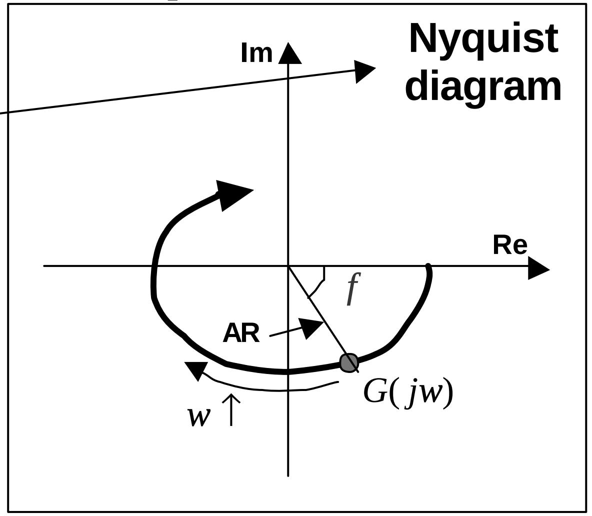
- **The frequency response, $G(jw)$ is complex as a function of frequency.**

$$G(jw) = \mathrm{Re}[G(jw)] + j\,\mathrm{Im}[G(jw)]$$

$$AR = |G(jw)| = \sqrt{\mathrm{Re}[G(jw)]^2 + \mathrm{Im}[G(jw)]^2}$$

$$f = \measuredangle G(jw) = \tan^{-1}\left(\mathrm{Im}[G(jw)] / \mathrm{Re}[G(jw)]\right)$$

**Bode plot**



- **Getting ultimate response**
  - **For a sinusoidal forcing function** $Y(s) = G(s)\dfrac{Aw}{s^2 + w^2}$
  - **Assume** $G(s)$ **has stable poles** $b_i$.

**Decayed out at large** $t$

$$Y(s) = G(s)\frac{Aw}{s^2+w^2} = \frac{a_1}{s+b_1} + \cdots + \frac{a_n}{s+b_n} + \frac{Cs+Dw}{s^2+w^2}$$

$$G(jw)Aw = Cjw + Dw \Rightarrow G(jw) = \frac{D}{A} + j\frac{C}{A} = R + jI$$

$$C = IA,\ D = RA \ \Rightarrow\ y_{ul} = A(I\cos wt + R\sin wt) = \hat{A}\sin(wt + f)$$

$$\therefore AR = \hat{A}/A = \sqrt{R^2 + I^2} = \left|G(jw)\right| \text{ and } f = \tan^{-1}(I/R) = \measuredangle G(jw)$$

  - **Without calculating transient response, the frequency response can be obtained directly from** $G(jw)$ **.**
  - **Unstable transfer function does not have a frequency response because a sinusoidal input produces an unstable output response.**

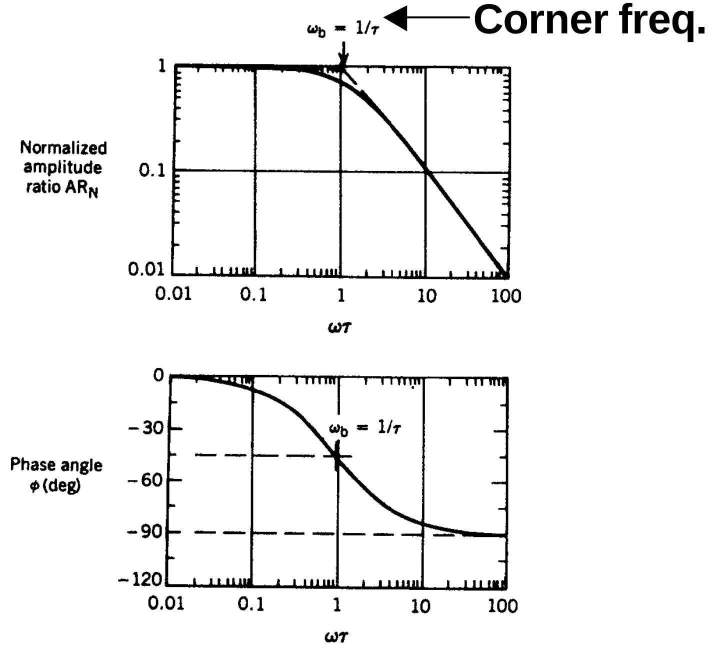
- **First-order process**

$$G(s)=\frac{K}{(t\,s+1)}$$

$$G(jw)=\frac{K}{(1+jwt)}=\frac{K}{(1+w^2t^2)}(1-jwt)$$

$$AR_N=\left|G(jw)\right|=\frac{1}{\sqrt{1+w^2t^2}}$$

$$f=\measuredangle G(jw)=-\tan^{-1}(wt)$$




- **Second-order process**

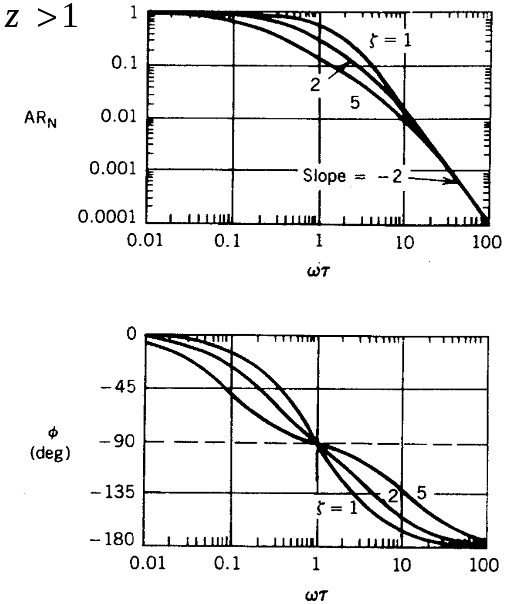
$$G(s)=\frac{K}{(t^2s^2+2zt\,s+1)}$$

$$G(jw)=\frac{K}{(1-t^2w^2)+2jztw}$$

$$AR=\left|G(jw)\right|=\frac{K}{\sqrt{(1-w^2t^2)^2+(2zwt)^2}}$$

$$f=\measuredangle G(jw)=\tan^{-1}\frac{\mathrm{Im}(G(jw))}{\mathrm{Re}(G(jw))}=-\tan^{-1}\frac{2zwt}{1-w^2t^2}$$

$z>1$



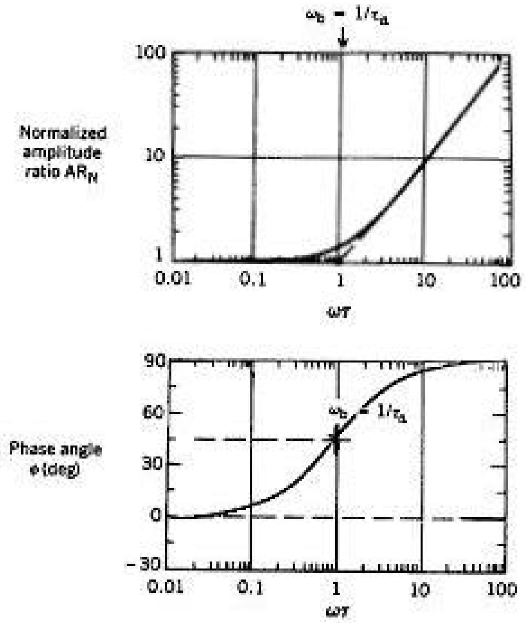
- **Process Zero (lead)**

$$G(s) = t_a s + 1$$

$$G(jw) = 1 + jwt_a$$

$$AR_N = |G(jw)| = \sqrt{1 + w^2 t_a^2}$$

$$f = \measuredangle G(jw) = \tan^{-1}(wt_a)$$



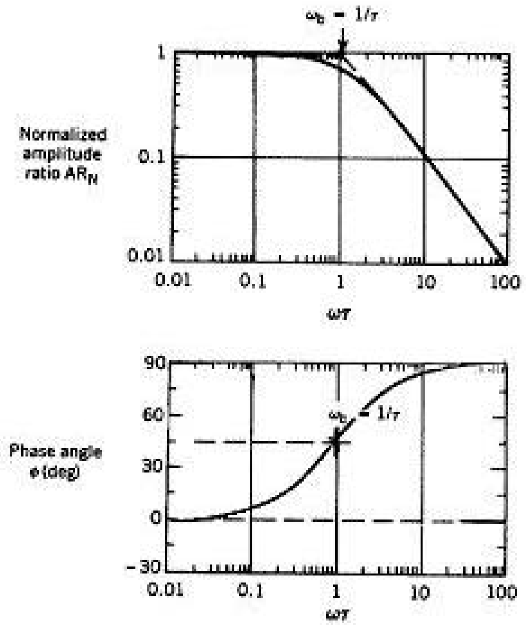

- **Unstable pole**

$$G(s) = \frac{1}{(-ts + 1)}$$

$$G(jw) = \frac{1}{1 - jtw} = \frac{1}{1 + t^2 w^2}(1 + jtw)$$

$$AR = |G(jw)| = \frac{1}{\sqrt{1 + w^2 t^2}}$$

$$f = \measuredangle G(jw) = \tan^{-1}\frac{\mathrm{Im}(G(jw))}{\mathrm{Re}(G(jw))} = \tan^{-1} wt$$

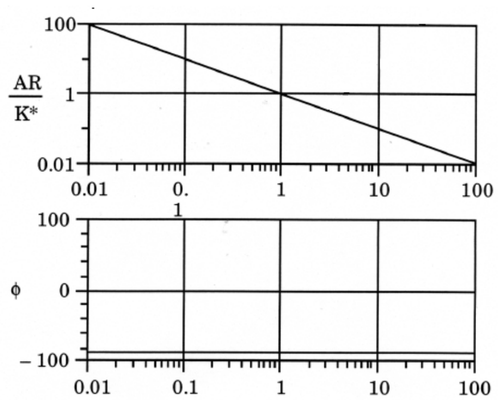


- **Integrating process**

$$G(s) = \frac{1}{As} \qquad G(jw) = \frac{1}{jAw} = -\frac{1}{Aw}j$$

$$AR_N = |G(jw)| = \frac{1}{Aw}$$

$$f = \measuredangle G(jw) = \tan^{-1}(-\frac{1}{0 \cdot w}) = -\frac{p}{2}$$

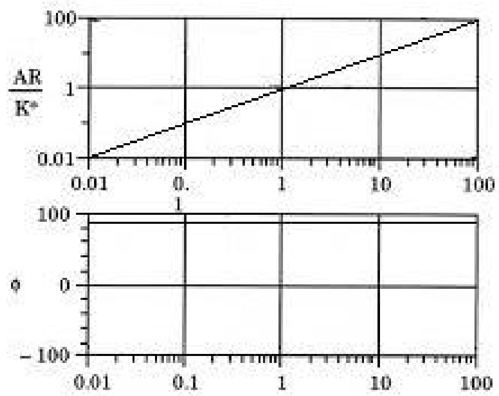



- **Differentiator**

$$G(s) = As \qquad G(jw) = jAw$$

$$AR_N = |G(jw)| = Aw$$

$$f = \measuredangle G(jw) = \tan^{-1}(\frac{1}{0 \cdot w}) = \frac{p}{2}$$



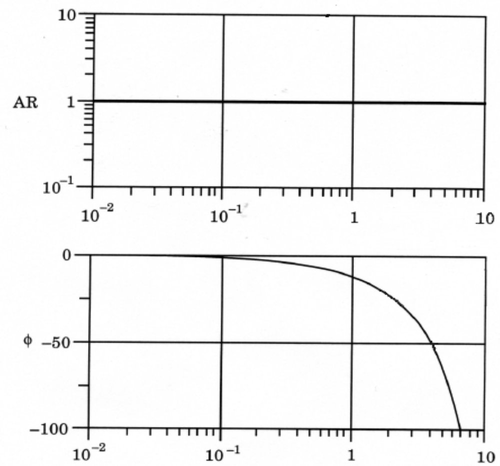

- **Pure delay process**

$$G(s) = e^{-qs}$$

$$G(jw) = e^{-jqw} = \cos qw - j\sin qw$$

$$AR = |G(jw)| = 1$$

$$f = \measuredangle G(jw) = -\tan^{-1}\tan qw = -qw$$



# SKETCHING BODE PLOT

$$G(s)=\frac{G_a(s)G_b(s)G_c(s)\cdots}{G_1(s)G_2(s)G_3(s)\cdots} \qquad G(jw)=\frac{G_a(jw)G_b(jw)G_c(jw)\cdots}{G_1(jw)G_2(jw)G_3(jw)\cdots}$$

$$|G(jw)|=\frac{|G_a(jw)||G_b(jw)||G_c(jw)|\cdots}{|G_1(jw)||G_2(jw)||G_3(jw)|\cdots}$$

$$\begin{aligned}\measuredangle G(jw)&=\measuredangle G_a(jw)+\measuredangle G_b(jw)+\measuredangle G_c(jw)+\cdots\\&\quad-\measuredangle G_1(jw)-\measuredangle G_2(jw)-\measuredangle G_3(jw)-\cdots\end{aligned}$$

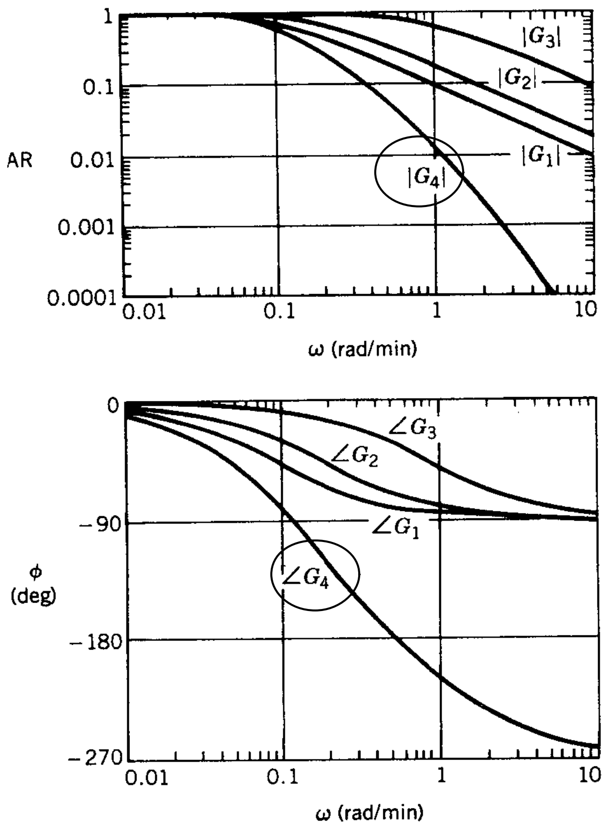
- **Bode diagram**
  - **AR vs. frequency in log-log plot**
  - **PA vs. frequency in semi-log plot**
  - **Useful for**
    - **Analysis of the response characteristics**
    - **Stability of the closed-loop system only for open-loop stable systems with phase angle curves exhibit a single critical frequency.**

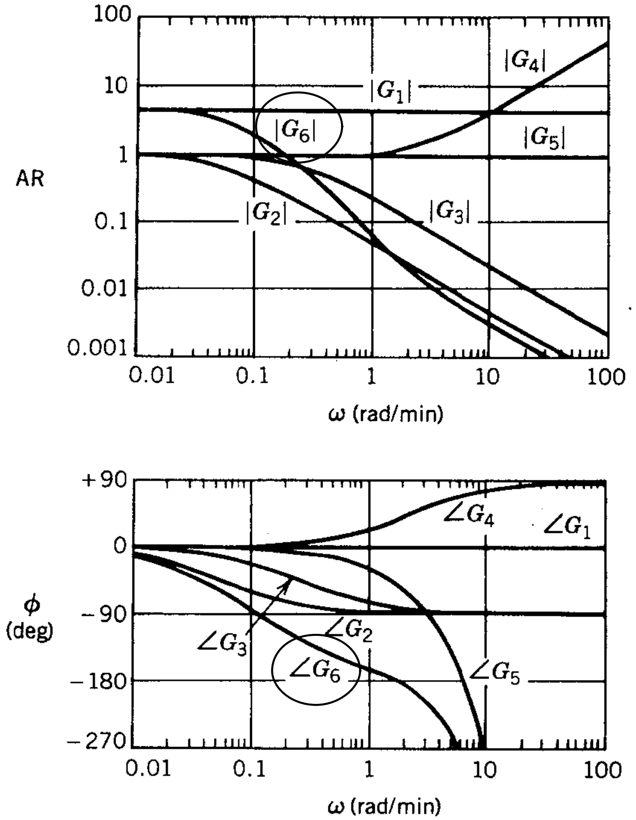
- **Amplitude Ratio on log-log plot**
  - Start from steady-state gain at $w$ =0. If $G_{OL}$ includes either integrator or differentiator it starts at $\infty$ or 0.
  - Each first-order lag (lead) adds to the slope –1 (+1) starting at the corner frequency.
  - Each integrator (differentiator) adds to the slope –1 (+1) starting at zero frequency.
  - A delays does not contribute to the AR plot.
- **Phase angle on semi-log plot**
  - Start from 0° or -180° at $w$ =0 depending on the sign of steady-state gain.
  - Each first-order lag (lead) adds 0° to phase angle at $w$ =0, adds -90° (+90°) to phase angle at $w = \infty$, and adds -45° (+45°) to phase angle at corner frequency.
  - Each integrator (differentiator) adds -90° (+90°) to the phase angle for all frequency.
  - A delay adds $-qw$ to phase angle depending on the frequency.

# Examples

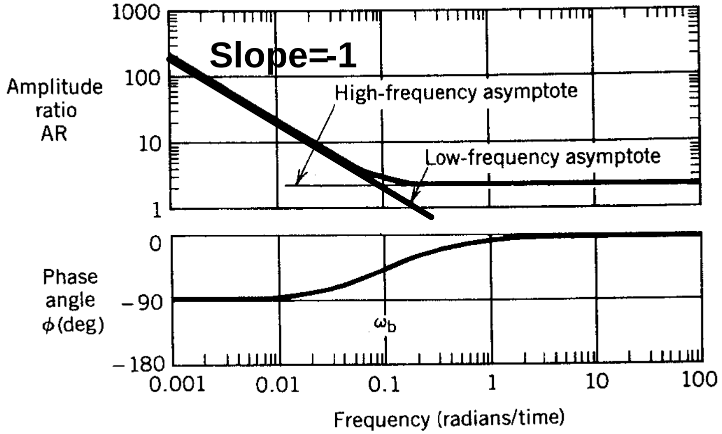
**1.** $$G(s)=\frac{K}{\underbrace{(10s+1)}_{G_1}\underbrace{(5s+1)}_{G_2}\underbrace{(s+1)}_{G_3}}$$

**2.** $$G(s)=\frac{\overbrace{5}^{G_1}\overbrace{(0.5s+1)}^{G_5}\overbrace{e^{-0.5s}}^{G_4}}{\underbrace{(20s+1)}_{G_2}\underbrace{(4s+1)}_{G_3}}$$



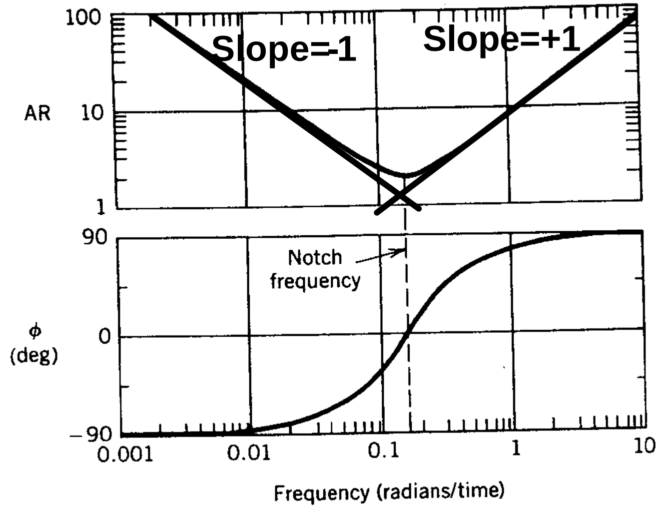



## 3. PI: $G(s) = K_C\left(1 + \frac{1}{t_I s}\right)$




$$w_b = 1/t_I \text{ at } f = -45°$$

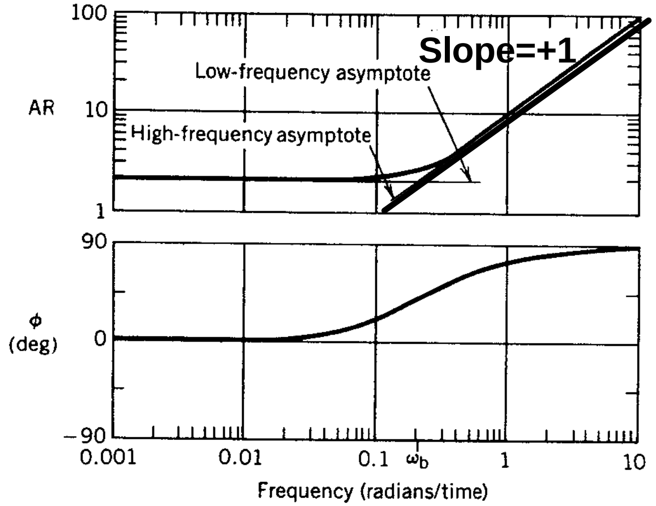
## 5. PID: $G(s) = K_C\left(1 + \frac{1}{t_I s} + t_D s\right)$




$$w_{Notch} = 1/\sqrt{t_I t_D} \text{ at } f = 0°$$

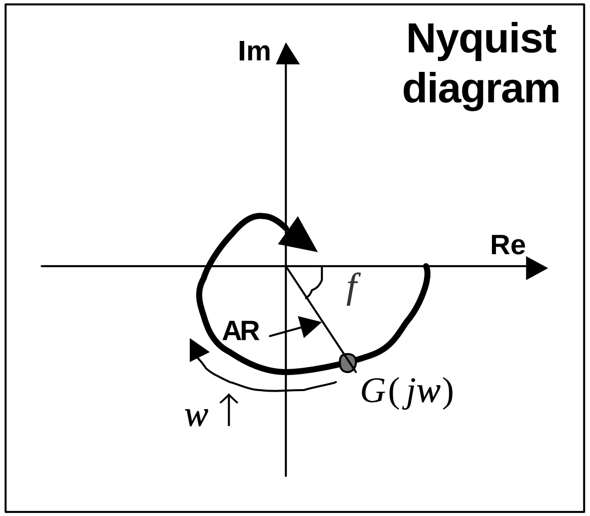
## 4. PD: $G(s) = K_C\left(1 + t_D s\right)$

$$w_b = 1/t_D \text{ at } f = -45°$$



# NYQUIST DIAGRAM

- **Alternative representation of frequency response**
- **Polar plot of $G(jw)$ ($w$ is implicit)**




$$G(jw) = \mathrm{Re}[G(jw)] + j\,\mathrm{Im}[G(jw)]$$

- **Compact (one plot)**
- **Wider applicability of stability analysis than Bode plot**
- **High frequency characteristics will be shrunk near the origin.**
  - **Inverse Nyquist diagram: polar plot of** $1/G(jw)$
- **Combination of different transfer function components is not easy as with Nyquist diagram as with Bode plot.**