Chap. 8. Vector Differential Calculus. Grad. Div. Curl

8.1. Vector Algebra in 2-Space and 3-Space

- Scalar: a quantity only with its magnitude; temperature, speed, mass, volume, …
- Vector: a quantity with its magnitude and its direction; velocity, acceleration, force, …
  (arrow & directed line segment)
  ↖ *Initial and termination point*

  Norm of $\underline{a}$: length of a vector $\underline{a}$. $|\underline{a}|$
  =1: unit vector

Equality of a Vectors: $\underline{a}=\underline{b}$: same length and direction.

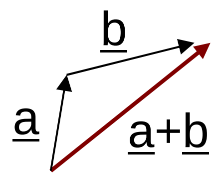
Components of a Vector: $P(x_1,y_1,z_1) \rightarrow Q(x_2,y_2,z_2)$ in Cartesian coordinates.

$$\underline{a} = \overrightarrow{PQ} = [x_2 - x_1, y_2 - y_1, z_2 - z_1] = [a_1, a_2, a_3]$$

Length in Terms of Components: $|\underline{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

Position Vector: from origin (0,0,0) → point A (x,y,z): $\underline{r}=[x,y,z]$

Vector Addition, Scalar Multiplication



(1) Addition: $\underline{a} + \underline{b} = [a_1 + b_1, a_2 + b_2, a_3 + b_3]$

$\underline{a} + \underline{b} = \underline{b} + \underline{a}$ $\quad$ $(\underline{u} + \underline{v}) + \underline{w} = \underline{u} + (\underline{v} + \underline{w})$

$\underline{a} + \underline{0} = \underline{0} + \underline{a} = \underline{a}$ $\quad$ $\underline{a} + (-\underline{a}) = \underline{0}$

(2) Multiplication: $c\underline{a} = [c\underline{a}_1, c\underline{a}_2, c\underline{a}_3]$

$c(\underline{a} + \underline{b}) = c\underline{a} + c\underline{b}$ $\quad$ $(c + k)\,\underline{a} = c\underline{a} + k\underline{a}$

$c(k\underline{a}) = ck\underline{a}$ $\quad$ $1\underline{a} = \underline{a}$ $\quad$ $0\underline{a} = \underline{0}$ $\quad$ $(-1)\underline{a} = -\underline{a}$

Unit Vectors: $\underline{i}$, $\underline{j}$, $\underline{k}$ $\quad$ $\underline{a} = [a_1, a_2, a_3] = a_1\underline{i} + a_2\underline{j} + a_3\underline{k}$

$\underline{i} = [1,0,0]$, $\underline{j} = [0,1,0]$, $\underline{k} = [0,0,1]$

8.2. Inner Product (Dot Product)

Definition: $\underline{a} \cdot \underline{b} = |\underline{a}||\underline{b}|\cos\gamma \quad \text{if } \underline{a} \neq \underline{0}, \underline{b} \neq \underline{0}$

$$\underline{a} \cdot \underline{b} = 0 \quad \text{if } \underline{a} = \underline{0} \text{ or } \underline{b} = \underline{0};\ \cos\gamma = 0$$

$$\underline{a} \cdot \underline{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 = \sum_{i=1}^{3} a_i b_i$$

$\underline{a} \cdot \underline{b} = 0$ ($\underline{a}$ is orthogonal to $\underline{b}$; $\underline{a}$, $\underline{b}$=orthogonal vectors)

Theorem 1:
The inner product of two nonzero vectors is zero iff these vectors are perpendicular.

Length and Angle in Terms of Inner Product:

$$\text{length of } \underline{a} : |\underline{a}| = \sqrt{\underline{a} \cdot \underline{a}}$$

$$\text{angle btw two vectors} : \cos\gamma = \frac{\underline{a} \cdot \underline{b}}{|\underline{a}||\underline{b}|} = \frac{\underline{a} \cdot \underline{b}}{\sqrt{\underline{a} \cdot \underline{a}}\sqrt{\underline{b} \cdot \underline{b}}}$$

Ex. 1) $\underline{a}$=[1,2,0], $\underline{b}$=[3,-2,1], angle btw $\underline{a}$ and $\underline{b}$?

General Properties of Inner Products:

$$[q_1\underline{a} + q_2\underline{b}] \cdot \underline{c} = q_1\underline{a} \cdot \underline{c} + q_2\underline{b} \cdot \underline{c} \qquad \underline{a} \cdot \underline{b} = \underline{b} \cdot \underline{a} \qquad \underline{a} \cdot \underline{a} \geq 0$$

$$[\underline{a} + \underline{b}] \cdot \underline{c} = \underline{a} \cdot \underline{c} + \underline{b} \cdot \underline{c}$$

$$|\underline{a} \cdot \underline{b}| \leq |\underline{a}||\underline{b}| \qquad \text{Schwarz inequality}$$

$$|\underline{a} + \underline{b}| \leq |\underline{a}| + |\underline{b}| \qquad \text{Triangle inequality}$$

$$|\underline{a} + \underline{b}|^2 + |\underline{a} - \underline{b}|^2 = 2\left(|\underline{a}|^2 + |\underline{b}|^2\right) \qquad \text{Parallelogram equality}$$

Derivation of $\underline{a}\cdot\underline{b}=a_1b_1+a_2b_2+a_3b_3$

$$\underline{a}=a_1\underline{i}+a_2\underline{j}+a_3\underline{k},\quad \underline{b}=b_1\underline{i}+b_2\underline{j}+b_3\underline{k}$$

$$\underline{i}\cdot\underline{j}=\delta_{ij}\qquad \begin{matrix}1 \text{ for } i=j\\ 0 \text{ for } i\neq j\end{matrix}$$

$$\Rightarrow \underline{a}\cdot\underline{b}=a_1b_1\underline{i}\cdot\underline{i}+a_1b_2\underline{i}\cdot\underline{j}+a_1b_3\underline{i}\cdot\underline{k}+\cdots+a_2b_3\underline{k}\cdot\underline{k}=a_1b_1+a_2b_2+a_3b_3$$

Application of Inner Products:

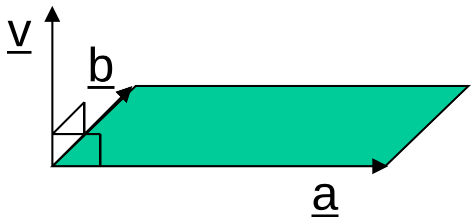
Ex. 2) Work

Ex. 3) Component of a force in a given direction

Ex. 5) Orthogonal straight lines in the plane

Ex. 6) Normal vector to a plane

8.3. Vector Product (Cross Product)



Definition: $\underline{v} = \underline{a} \times \underline{b}, \quad \text{length}: |\underline{v}| = |\underline{a}|\,|\underline{b}| \sin\gamma$

In components: $\underline{v} = [v_1, v_2, v_3]$

$= [a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1]$

Right-handed triple of vectors $\underline{a}$, $\underline{b}$, $\underline{v}$

$\underline{i}$, $\underline{j}$, $\underline{k}$ form a right-handed triple in the positive directions

How to memorize above formula:

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$\underline{i} \times \underline{j} = \sum_{k=1}^{3} \varepsilon_{ijk} \underline{k} \qquad \varepsilon_{ijk} = +1 \text{ if } ijk = 123, 231, 312$$

$$\varepsilon_{ijk} = -1 \text{ if } ijk = 321, 132, 213$$

$$\varepsilon_{ijk} = 0 \quad \text{if any two indices are alike}$$

General Properties of Vector Products:

$$(q\underline{a}) \times \underline{b} = q(\underline{a} \times \underline{b}) = \underline{a} \times (q\underline{b}) \qquad \underline{a} \times (\underline{b} + \underline{c}) = (\underline{a} \times \underline{b}) + (\underline{a} \times \underline{c})$$

$$(\underline{a} + \underline{b}) \times \underline{c} = (\underline{a} \times \underline{c}) + (\underline{b} \times \underline{c})$$

$$\underline{a} \times \underline{c} \neq \underline{c} \times \underline{a} \quad (\because \underline{i} \times \underline{j} \neq \underline{j} \times \underline{i}) \qquad \underline{a} \times (\underline{b} \times \underline{c}) \neq (\underline{a} \times \underline{b}) \times \underline{c} \quad (\because (\underline{i} \times \underline{i}) \times \underline{j} \neq \underline{i} \times (\underline{i} \times \underline{j}))$$

Typical Applications of Vector Products

Ex. 4) Moment of a force (I)

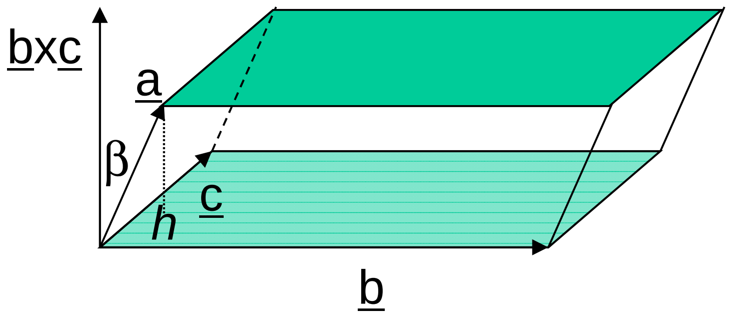
Ex. 5) Moment of a force (II)

Ex. 6) Velocity of a rotating body

Scalar Triple Product:

$$\underline{a} = [a_1, a_2, a_3],\ \underline{b} = [b_1, b_2, b_3],\ \underline{c} = [c_1, c_2, c_3]$$

$$(\underline{a}\ \underline{b}\ \underline{c}) = \underline{a} \cdot (\underline{b} \times \underline{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$



Volume of the parallelepiped with $\underline{a}$, $\underline{b}$, $\underline{c}$

$$(k\underline{a}\ \underline{b}\ \underline{c}) = k(\underline{a}\ \underline{b}\ \underline{c}) \qquad \underline{a} \cdot (\underline{b} \times \underline{c}) = (\underline{a} \times \underline{b}) \cdot \underline{c} = \underline{c} \cdot (\underline{a} \times \underline{b})$$

see matrix expression

Theorem 1: Three vectors form a linearly independent set iff their scalar triple product is not zero.

8.4. Vector and Scalar Functions and Fields. Derivatives

- Two kinds of functions

(1) Vector functions: $\underline{v} = \underline{v}(p) = [v_1(p), v_2(p), v_3(p)]$ depending on the point p in space.

→ a vector field

In Cartesian coordinates, $\underline{v}(x, y, z) = [v_1(x, y, z), v_2(x, y, z), v_3(x, y, z)]$

(2) Scalar functions: $f = f(p)$ depending on p. → a scalar field

Ex. 1) Scalar function (Euclidean distance in space)

Ex. 2, 3) Vector function (Velocity field, force field)

Vector Calculus

- Basic concepts of vector calculus

(1) Convergence: $\lim_{n\to\infty} \left|\underline{a}_n - \underline{a}\right| = 0$ or $\lim_{n\to\infty} \underline{a}_n = \underline{a}$ ($\underline{a}$: limit vector)

$$\lim_{t\to t_0} \left|\underline{v}(t) - \underline{u}\right| = 0 \text{ or } \lim_{t\to t_0} \underline{v}(t) = \underline{u}$$

(*vector function $\underline{v}$ of a real variable t has limit $\underline{u}$*)

(2) Continuity: $\lim_{t\to t_0} \underline{v}(t) = \underline{v}(t_0)$ vector function $\underline{v}$(t) is continuous at $t=t_0$.

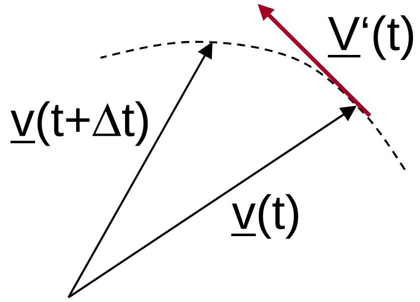
$$\underline{v}(t) = [v_1(t), v_2(t), v_3(t)] = v_1(t)\underline{i} + v_2(t)\underline{j} + v_3(t)\underline{k}$$

three components are continuous at t_0.

Derivative of a vector function

Definition:

$$\underline{v}'(t) = \lim_{\Delta t \to 0} \frac{\underline{v}(t+\Delta t) - \underline{v}(t)}{\Delta t}$$



$$\underline{v}'(t) = [v_1'(t), v_2'(t), v_3'(t)]$$

Derivative $\underline{v}'(t)$ is obtained by differentiating each component separately.

$$(c\underline{v}') = c\underline{v}' \qquad (\underline{v}+\underline{u})' = \underline{v}' + \underline{u}' \qquad (\underline{u}\cdot\underline{v})' = \underline{u}'\cdot\underline{v} + \underline{u}\cdot\underline{v}'$$

$$(\underline{u}\times\underline{v})' = \underline{u}'\times\underline{v} + \underline{u}\times\underline{v}' \qquad (\underline{u}\ \ \underline{v}\ \ \underline{w})' = (\underline{u}'\ \ \underline{v}\ \ \underline{w}) + (\underline{u}\ \ \underline{v}'\ \ \underline{w}) + (\underline{u}\ \ \underline{v}\ \ \underline{w}')$$

Partial Derivatives of a Vector Function

$$\underline{v}(t) = [v_1, v_2, v_3] = v_1\underline{i} + v_2\underline{j} + v_3\underline{k}$$ differentiable functions of n variables, $t_1, \ldots, t_n$.

Partial derivative of $\underline{v}$:

$$\frac{\partial \underline{v}}{\partial t_l} = \frac{\partial v_1}{\partial t_l}\underline{i} + \frac{\partial v_2}{\partial t_l}\underline{j} + \frac{\partial v_3}{\partial t_l}\underline{k} \qquad \frac{\partial^2 \underline{v}}{\partial t_l \partial t_m} = \frac{\partial^2 v_1}{\partial t_l \partial t_m}\underline{i} + \frac{\partial^2 v_2}{\partial t_l \partial t_m}\underline{j} + \frac{\partial^2 v_3}{\partial t_l \partial t_m}\underline{k}$$