9.2. Line Integrals Independent of Path

$$\int_C \underline{F}(\underline{r})\cdot d\underline{r} = \int_C \left(F_1 dx + F_2 dy + F_3 dz\right)$$

$$= \int_a^b \left(F_1 x' + F_2 y' + F_3 z'\right) dt$$

Theorem 1: **Independence of path**

Above line integral with continuous F_1, F_2, F_3 in D in space is *independent* of path in D iff $F=[F_1, F_2, F_3]$ is the gradient of some function f (*potential*) in D.

$$\underline{F} = \underline{\nabla} f; \quad F_1 = \frac{\partial f}{\partial x}, \ F_2 = \frac{\partial f}{\partial y}, \ F_3 = \frac{\partial f}{\partial z}$$

Ex. 1) $\int_C \underline{F}(\underline{r})\cdot d\underline{r} = \int_C \left(F_1 dx + F_2 dy + F_3 dz\right) = \int_C \left(2x dx + 2y dy + 4z dz\right)$ A(0,0,0) → B(2,2,2)

$$f = x^2 + y^2 + 2z^2$$

$$\underline{r}(t) = x(t)\underline{i} + y(t)\underline{j} + z(t)\underline{k}, \quad a \le t \le b$$

$$\int_A^B \left(F_1 dx + F_2 dy + F_3 dz\right) = \int_A^B \left(\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz\right)$$

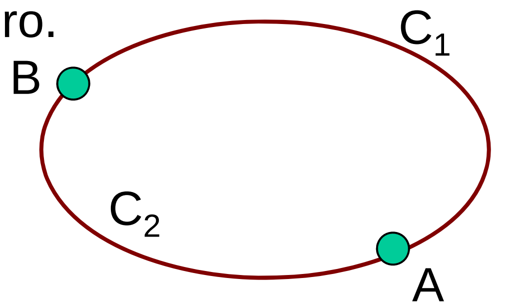
$$= \int_a^b \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}\right) dt = \int_a^b \left(\frac{df}{dt}\right) dt = f(B) - f(A), \qquad \underline{F} = \underline{\nabla} f$$

Ex. 2) $I = \int_C \left(3x^2 dx + 2yzdy + y^2 dz\right), \qquad A(0,1,2) \rightarrow B(1,-1,7)$

$$f(x,y,z) = x^3 + y^2 z \;\Rightarrow\; I = f(1,-1,7) - f(0,1,2) = 6$$

Integration Around Closed Curves and Independence of Path

Theorem 2: The integral is *independent* of path in a domain D iff its value around every closed path in D is zero.



Exactness and Independence of Path

Path independence ~ gradient
integration around closed curves
exactness of the differential form $F_1 dx + F_2 dy + F_3 dz$

- $F_1 dx + F_2 dy + F_3 dz$: exact in D if $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$ or $df = F_1 dx + F_2 dy + F_3 dz$

- Above form is exact iff there is a differentiable function f(x,y,z) in D such that

$$F_1 = \frac{\partial f}{\partial x},\; F_2 = \frac{\partial f}{\partial y},\; F_3 = \frac{\partial f}{\partial z} \quad \Rightarrow \underline{F} = \underline{\nabla} f$$

→ A line integral is *independent* of path in D iff the differential form, $F_1 dx + F_2 dy + F_3 dz$ has continuous F_1, F_2, F_3 and is exact in D.

Theorem 3: $\int_C \underline{F}(\underline{r}) \cdot d\underline{r} = \int_C (F_1 dx + F_2 dy + F_3 dz)$

(a) If line integral is *independent* of path in D and thus the differential form $F_1 dx + F_2 dy + F_3 dz$ is exact in D,

$$\Rightarrow \ \underline{\nabla} \times \underline{F} = \underline{0} \qquad \frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$$

$$\underline{\nabla} \times \underline{F} = \underline{\nabla} \times (\underline{\nabla} f) = \underline{0}$$

(b) Above relation holds in D and D is simply connected, then line integral is *indep.* of path in D.

- A domain is simply connected if every closed curve in D can be continuously shrunk to any point in D without leaving D.

Ex) *simply connected: interior of a sphere or a cube, domain btw two concentric spheres…*
not simply connected: a torus…

Ex. 3) $I = \int_C \left(2xyz^2 dx + (x^2 z^2 + z\cos yz)dy + (2x^2 yz + y\cos yz)dz\right)$

$$\Rightarrow f(x, y, z) = x^2 yz^2 + \sin yz$$

Ex. 4) On the assumption of simple connectedness (see textbook)

9.3. Double Integrals

- Double Integrals of f(x,y) over the region R

$$\iint_R f(x,y)\,dxdy \quad \text{or} \quad \iint_R f(x,y)\,dA$$

- Some properties of double integrals

$$\iint_R kf\ dxdy = k\iint_R f\ dxdy$$

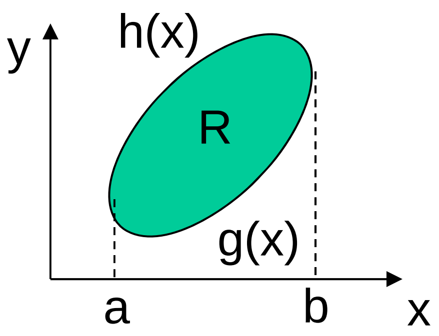
$$\iint_R (f+g)\ dxdy = \iint_R f\ dxdy + \iint_R g\ dxdy \qquad \iint_R f\ dxdy = \iint_{R_1} f\ dxdy + \iint_{R_2} f\ dxdy$$

- Mean value theorem: $\iint_R f(x,y)\ dxdy = f(x_0,y_0)A$

Evaluation of Double Integrals

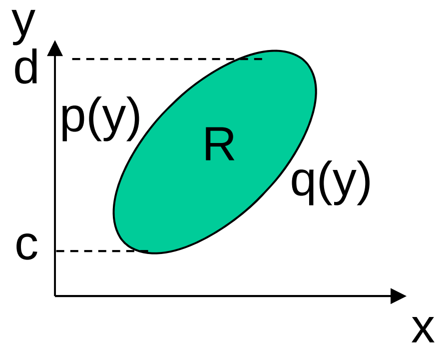
$$a \le x \le b,\ \ g(x) \le y \le h(x)$$

$$\iint_R f(x,y)\,dxdy = \int_a^b \left[\int_{g(x)}^{h(x)} f(x,y)\,dy\right]dx$$



$$c \le y \le d,\ \ p(y) \le x \le q(y)$$

$$\iint_R f(x,y)\,dxdy = \int_c^d \left[\int_{p(y)}^{q(y)} f(x,y)\,dx\right]dy$$



Change of Variables in Double Integrals. Jacobian

$$\int_a^b f(x)dx = \int_\alpha^\beta f(x(u))\frac{dx}{du}du$$

$$\iint_R f(x,y)dxdy = \iint_{R^*} f(x(u,v),y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right|dudv$$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

Ex) $x=r\cos\theta$, $y=r\sin\theta$

$J=r$

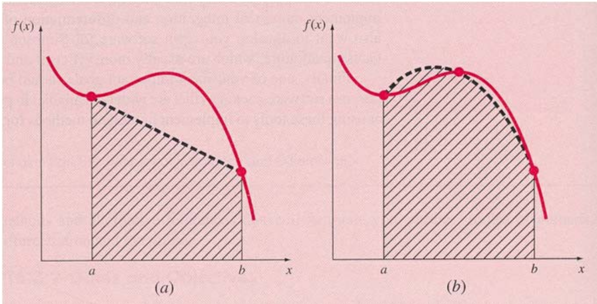
$$\iint_R f(x,y)dxdy = \iint_{R^*} f(r\cos\theta, r\sin\theta)r\,drd\theta$$

적분의 수치해석 (simple case 참고)

Most common numerical integration schemes
- replacing a complicated function or tabulated data with an easy integrable approx. func.

$$I = \int_a^b f(x)dx \cong \int_a^b f_n(x)dx$$ **($f_n(x)$=a polynomial func.,** $f_n(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$ **)**

Straight line fn.



Parabola fn.

Trapezoidal Rule

$$I = \int_a^b f(x)dx \cong \int_a^b f_1(x)dx$$ **, first-order polynomial:** $f_1(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a)$

$$I = (b-a)\frac{f(a) + f(b)}{2}$$

By extending above formulation, $I = \int_a^b f(x)dx = \frac{h}{2}\left(f_0 + 2f_1 + 2f_2 + \cdots + 2f_{n-1} + f_n\right)$

$$\Rightarrow \quad I = \frac{h}{2}\left(f_0 + 2\sum_{j=1}^{n-1} f_j + f_n\right) \quad \left(f_0 = f(a),\ f_n = f(b)\right)$$

Multiple Integrals:

$$I = \int_a^b \int_{c(x)}^{d(x)} f(x, y)\, dy\, dx$$

Simple case: use n equally spaced subintervals in x

$$g(a) = \int_{c(a)}^{d(a)} f(a, y)\, dy$$

$$g(a+\Delta x) = \int_{c(a+\Delta x)}^{d(a+\Delta x)} f(a+\Delta x, y)\, dy$$

$$g(b) = \int_{c(b)}^{d(b)} f(b, y)\, dy$$

Trapezoidal rule: $\mathrm{I} \cong \frac{\Delta \mathrm{x}}{2}\left[\mathrm{g(a)} + \mathrm{g(b)} + 2\sum_{\mathrm{j=1}}^{\mathrm{n-1}} \mathrm{g(a + j\Delta x)}\right]$

9.4. Green Theorem in the Plane

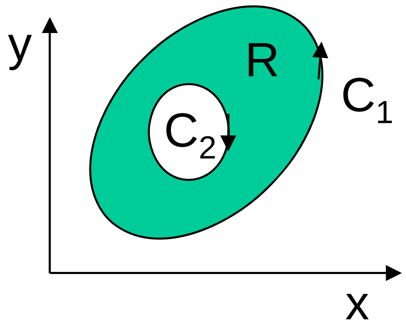
- *Double integrals* over a plane region ↔ *Line integrals* over the boundary of the region (easy evaluation of integrals)

Theorem 1: **Green's theorem in the plane (transformation btw double and line integrals)**

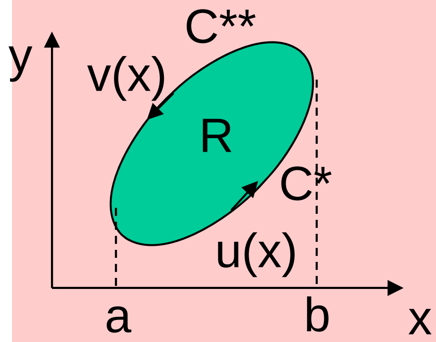
R: a closed bounded region in xy plane

Boundary C

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dxdy = \oint_C (F_1 dx + F_2 dy) \qquad \iint_R (\underline{\nabla} \times \underline{F}) \cdot \underline{k}\, dxdy = \oint_C \underline{F} \cdot d\underline{r} \quad (\underline{F} = F_1 \underline{i} + F_2 \underline{j})$$



integrated along the entire boundary C of R such that R is on the left in the direction of integration



Ex. 1)

$$a \le x \le b, \;\; u(x) \le y \le v(x)$$

$$\iint_R \frac{\partial F_1}{\partial y} dxdy = \int_a^b \left[\int_{u(x)}^{v(x)} \frac{\partial F_1}{\partial y} dy \right] dx$$

$$\iint_R \frac{\partial F_1}{\partial y} dxdy = \int_a^b [F_1(x, v(x)) - F_1(x, u(x))] dx$$

$$= \int_a^b F_1(x, v(x)) dx - \int_a^b F_1(x, u(x)) dx$$

$$= -\int_b^a F_1(x, v(x)) dx - \int_a^b F_1(x, u(x)) dx = -\oint_C F_1(x, y) dx$$