Runge-Kutta 방법을 이용한 상미분 방정식의 풀이

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- Runge-Kutta 수학적 유도

f_x , f_y 의 편미분 계산을 하지 않고 $y(x_1)$의 근사값을 구하자.

$$\left.\begin{aligned} x &= x_0 + \varepsilon_0 \\ y &= y_0 + \mu_0 \end{aligned}\right\}$$

$f(x_0+\varepsilon_0, y_0+\mu\ \varepsilon_0)$를 (x_0, y_0)에 대하여 2변수 테일러 급수를 전개를 하면,

$$f(x_0+\varepsilon_0, y_0+\mu_0) = f(x_0, y_0) + \varepsilon_0 f_x(x_0, y_0) + \mu_0 f_y(x_0, y_0) + 1/2\{\varepsilon_0^2 f_{xx}(\varepsilon, \mu) + 2\mu_0\varepsilon_0 f_{xy}(\varepsilon, \mu) + \mu_0^2 f_{xy}(\varepsilon, \mu)\} \quad (12\text{-}11)$$

$$x_0 \le \varepsilon \le x_0 + \varepsilon_0, \qquad y_0 \le \mu \le y_0 + \mu_0$$

$\varepsilon_0 \equiv ph$, $\mu_0 = qhf(x_0, y_0)$ (p, q : integer)를 (12-11)식에 대입하면

$$f(x_0+ph, y_0+qh f(x_0, y_0))$$
$$\cong f(x_0, y_0) + ph f_x(x_0, y_0) = qh f_y(x_0, y_0) f(x_0, y_0)$$

$$W(x_0, y_0, h) \equiv a f(x_0, y_0) + b f(x_0, + ph, y_0 + qh f(x_0, y_0)) \quad (12\text{-}12)$$

$$\begin{aligned} &= a f(x_0, y_0) + b f(x_0, y_0) + bphf_x(x_0, y_0)) + bqhf_y(x_0, y_0) f(x_0, y_0) \\ &= (a+b) f(x_0, y_0) + hb\{p f_x(x_0, y_0) + q f_y(x_0, y_0) f(x_0, y_0)\} \end{aligned} \quad (12\text{-}13)$$

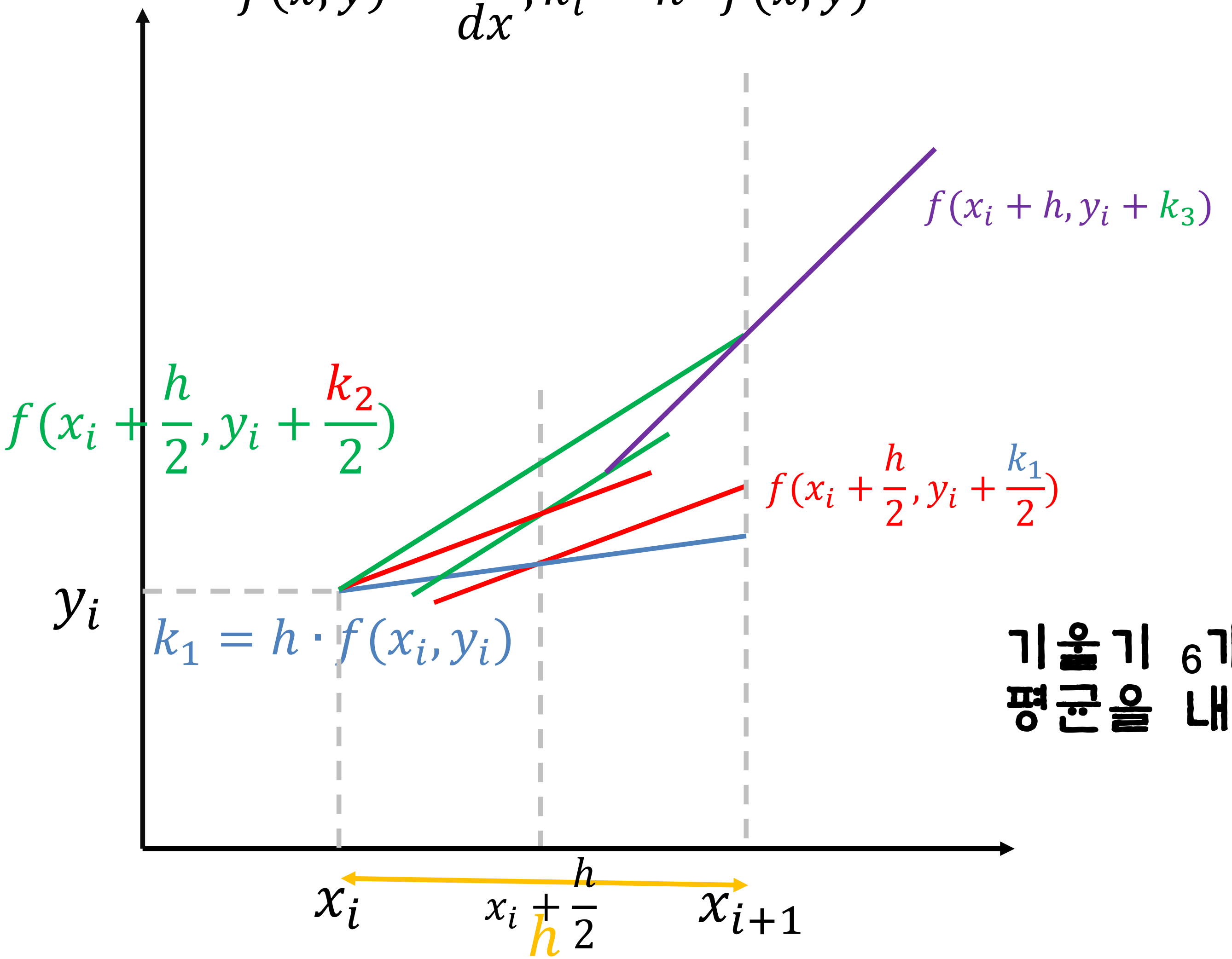
식 (12-9), (12-13)의 계수 비교를 하면,

$$a+b=1, \quad bp = bq = 1/2 \quad (12\text{-}14)$$

식 (12-12)로 부터

$$y_1 = y_0 + ah f(x_0, y_0) + bhf(x_0+hp, y_0+qhf(x_0, y_0)) \quad (12\text{-}15)$$

$$f(x,y) = \frac{dy}{dx}, k_i = h \cdot f(x,y)$$



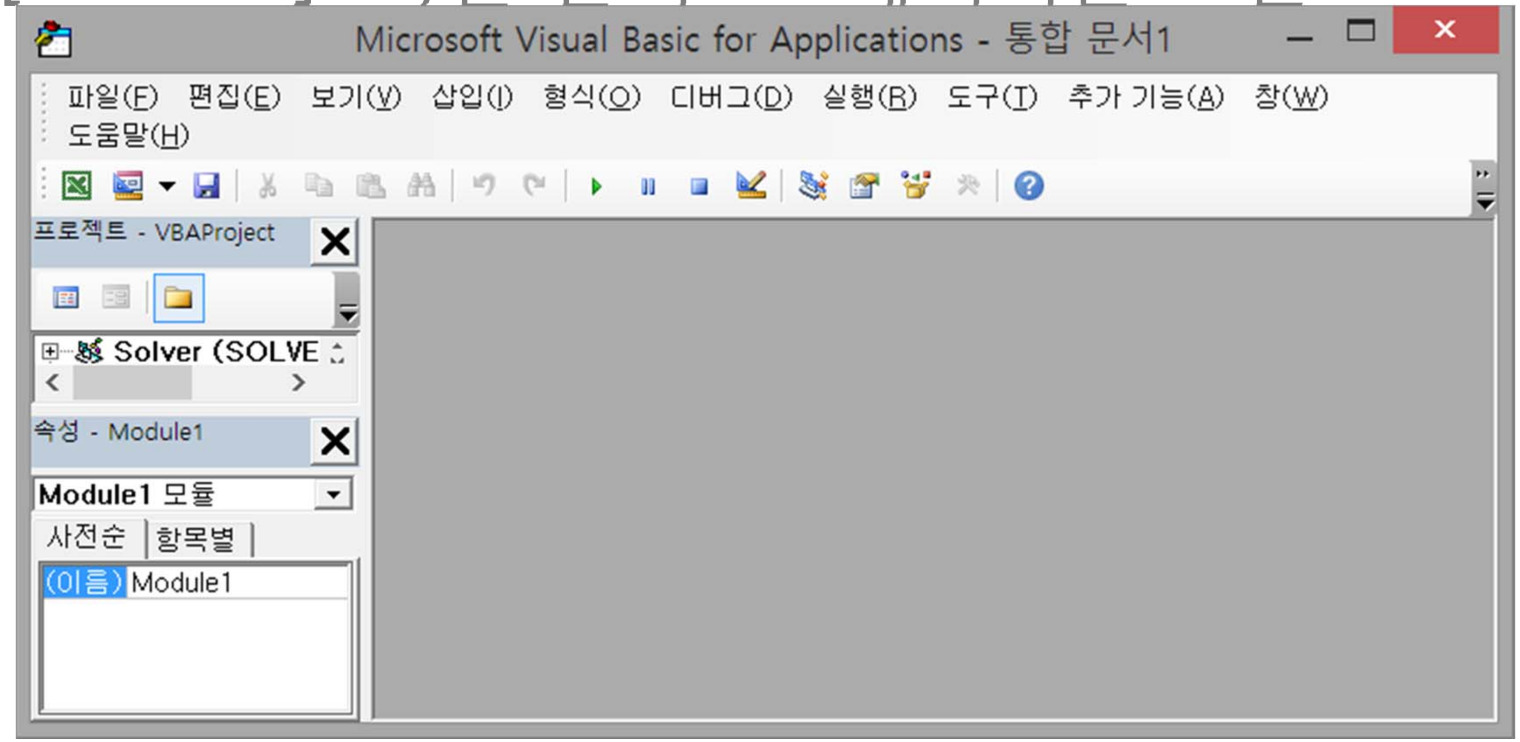
기울기 6개를 가지고
평균을 내서 구함!

Runge-Kutta 2nd order		$y_{i+1} = y_i + (1-b)k_1 + k_2 \quad (b \neq 0)$
		$k_1 = h \cdot f(x_i, y_i)$ $k_2 = h \cdot f\left(x_i + \frac{h}{2b}, y_i + \frac{k_1}{2b}\right)$
Runge-Kutta 3rd order		$y_{i+1} = y_i + \frac{1}{6}(k_1 + 4k_2 + k_3)$
		$k_1 = h \cdot f(x_i, y_i)$ $k_2 = h \cdot f\left(x_i + \frac{h}{2}, y_i + \frac{k_1}{2}\right)$ $k_3 = h \cdot f(x_i + h, y_i - 2k_1 + 2k_2)$
Runge-Kutta 4th order		$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$
		$k_1 = h \cdot f(x_i, y_i)$ $k_2 = h \cdot f\left(x_i + \frac{h}{2}, y_i + \frac{k_1}{2}\right)$ $k_3 = h \cdot f\left(x_i + \frac{h}{2}, y_i + \frac{k_2}{2}\right)$ $k_4 = h \cdot f(x_i + h, y_i + k_3)$
$f(x, y) = \frac{dy}{dx}$		

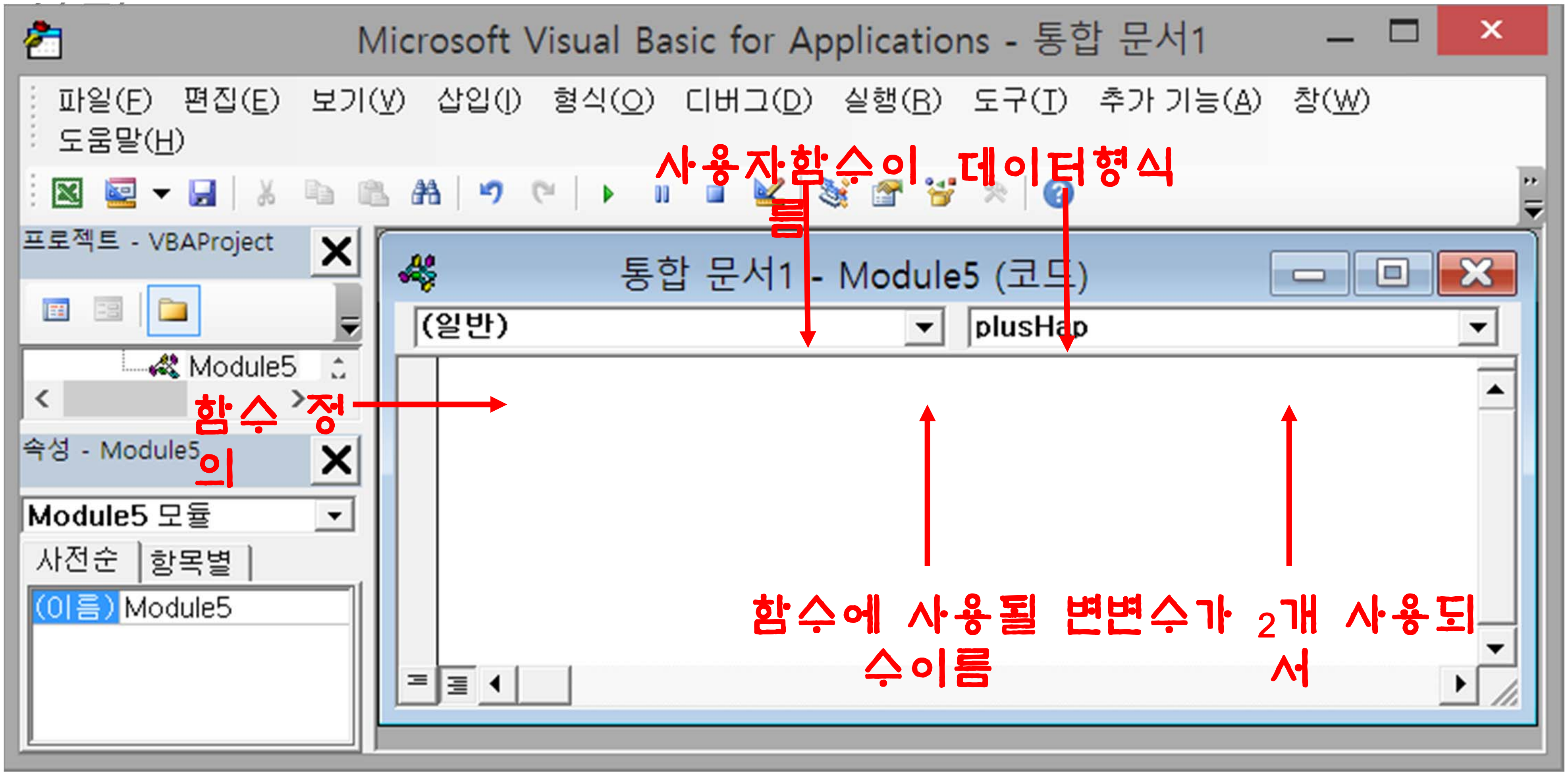
Solutions of ODEs by using the Visual Basic for Applications

- Building User-defined functions in VBA

[Alt+F11] key를 눌러 VB 에디터를 호출



메뉴의 [삽입>모듈]로 모듈을 만든 후, 다음을 입력



- Solutions using by the 2nd order RK

1. 셀 [h] 지정

2. [=h*plusHap(C8, E8)] 입력

3. [=h*plusHap(C8+h/2, E8+F8/2)] 입력

4. [=E8+G8] 입력

5. [=D8] 입력

	A	B	C	D	E	F	G
1			Runge-Kutta Method			$y_{i+1}=y_i+(1-b)k_1+k_2$	
2			dy/dx=x+y , y(0)=0				
3		2nd order	h =	0.1			
4			b =	1		$(y_{i+1}=y_i+k_2)$	
7		i	x_i	y_{i+1}	y_i	$k_1 = hf(x_i, y_i)$	$k_2 = hf(x_i+h/2, y_i+k_i/2)$
8		0	0.00	0.0050000	0.0000000	0.0000000	0.0050000
9		1	0.10		0.0050000		
10		2	0.20				
11		3	0.30				
12		4	0.40				
13		5	0.50				
14		6	0.60				
15		7	0.70				
16		8	0.80				
17		9	0.90				
18		10	1.00				

자동 채우기로 k_1, k_2, y_{i+1} 순으로 자동채움

	A	B	C	D	E	F	G
1			Runge-Kutta Method			$y_{i+1}=y_i+(1-b)k_1+k_2$	
2			dy/dx=x+y , y(0)=0				
3		2nd order	h =	0.1			
4			b =	1		$(y_{i+1}=y_i+k_2)$	
7		i	x_i	y_{i+1}	y_i	$k_1 = hf(x_i, y_i)$	$k_2 = hf(x_i+h/2, y_i+k_i/2)$
8		0	0.00	0.0050000	0.0000000	0.0000000	0.0050000
9		1	0.10		0.0050000	0.0105000	0.0160250
10		2	0.20				
11		3	0.30	3		1	2
12		4	0.40				
13		5	0.50				
14		6	0.60				
15		7	0.70				
16		8	0.80				
17		9	0.90				
18		10	1.00				

4개의 셀을 같이 선택한 후 자동채움

	A	B	C	D	E	F	G
1			Runge-Kutta Method			$y_{i+1}=y_i+(1-b)k_1+k_2$	
2			dy/dx=x+y , y(0)=0				
3		2nd order	h =	0.1			
4			b =	1		($y_{i+1}=y_i+k_2$)	
7		i	x_i	y_{i+1}	y_i	$k_1 = hf(x_i, y_i)$	$k_2 = hf(x_i+h/2, y_i+k_i/2)$
8		0	0.00	0.0050000	0.0000000	0.0000000	0.0050000
9		1	0.10	0.0210250	0.0050000	0.0105000	0.0160250
10		2	0.20				
11		3	0.30				
12		4	0.40				
13		5	0.50				
14		6	0.60				
15		7	0.70				
16		8	0.80				
17		9	0.90				
18		10	1.00				

	A	B	C	D	E	F	G
1			Runge-Kutta Method			$y_{i+1}=y_i+(1-b)k_1+k_2$	
2			dy/dx=x+y , y(0)=0				
3		2nd order	h =	0.1			
4			b =	1		$(y_{i+1}=y_i+k_2)$	
7		i	x_i	y_{i+1}	y_i	$k_1 = hf(x_i, y_i)$	$k_2 = hf(x_i+h/2, y_i+k_i/2)$
8		0	0.00	0.0050000	0.0000000	0.0000000	0.0050000
9		1	0.10	0.0210250	0.0050000	0.0105000	0.0160250
10		2	0.20	0.0492326	0.0210250	0.0221025	0.0282076
11		3	0.30	0.0909021	0.0492326	0.0349233	0.0416694
12		4	0.40	0.1474468	0.0909021	0.0490902	0.0565447
13		5	0.50	0.2204287	0.1474468	0.0647447	0.0729819
14		6	0.60	0.3115737	0.2204287	0.0820429	0.0911450
15		7	0.70	0.4227889	0.3115737	0.1011574	0.1112152
16		8	0.80	0.5561818	0.4227889	0.1222789	0.1333928
17		9	0.90	0.7140808	0.5561818	0.1456182	0.1578991
18		10	1.00				

- Solutions using by the 4th order RK

1. 셀 [h] 지정

B	C	D	E	F	G	H	I
	Runge-Kutta Method		$y_{i+1}=y_i+1/6(k_1+2k_2+2k_3+k_4)$				
	dy/dx=x+y , y(0)=0						
4th order	h =	0.1					
i	x_i	y_{i+1}	y_i	$k_1 = hf(x_i, y_i)$	$k_2 = hf(x_i+h/2, y_i+k_1/2)$	$k_3 = hf(x_i+h/2, y_i+k_2/2)$	$k_4=hf(x_i+h, y_i+k_3)$
0	0.00	0.0051708	0.0000000	0.0000000	0.0050000	0.0052500	0.0105250
1	0.10		0.0051708				
2	0.20						
3	0.30						
4	0.40						
5	0.50						
6	0.60						
7	0.70						
8	0.80						
9	0.90						
10	1.00						

2. [=h*plusHap(C8, E8)]

3. [=h*plusHap(C8+h/2, E8+F8/2)]

4. [=h*plusHap(C8+h/2, E8+G8/2)]

5. [=h*plusHap(C8+h, E8+H8)]

6. [=E8+(F8+2*G8+2*H8+I8)/6]

7. [=D8]

B	C	D	E	F	G	H	I
	Runge-Kutta Method		$y_{i+1}=y_i+1/6(k_1+2k_2+2k_3+k_4)$				
	$dy/dx=x+y$, $y(0)=0$						
4th order	h =	0.1					
i	x_i	y_{i+1}	y_i	$k_1 = hf(x_i, y_i)$	$k_2 = hf(x_i+h/2, y_i+k_1/2)$	$k_3 = hf(x_i+h/2, y_i+k_2/2)$	$k_4=hf(x_i+h,y_i+k_3)$
0	0.00	0.0051708	0.0000000	0.0000000	0.0050000	0.0052500	0.0105250
1	0.10	0.0214026	0.0051708	0.0105171	0.0160429	0.0163192	0.0221490
2	0.20						
3	0.30						
4	0.40						
5	0.50						
6	0.60						
7	0.70						
8	0.80						
9	0.90						
10	1.00						

B	C	D	E	F	G	H	I
	Runge-Kutta Method		$y_{i+1}=y_i+1/6(k_1+2k_2+2k_3+k_4)$				
	dy/dx=x+y , y(0)=0						
4th order	h =	0.1					
i	x_i	y_{i+1}	y_i	$k_1 = hf(x_i, y_i)$	$k_2 = hf(x_i+h/2, y_i+k_1/2)$	$k_3 = hf(x_i+h/2, y_i+k_2/2)$	$k_4=hf(x_i+h, y_i+k_3)$
0	0.00	0.0051708	0.0000000	0.0000000	0.0050000	0.0052500	0.0105250
1	0.10	0.0214026	0.0051708	0.0105171	0.0160429	0.0163192	0.0221490
2	0.20	0.0498585	0.0214026	0.0221403	0.0282473	0.0285526	0.0349955
3	0.30	0.0918242	0.0498585	0.0349859	0.0417351	0.0420726	0.0491931
4	0.40	0.1487206	0.0918242	0.0491824	0.0566415	0.0570145	0.0648839
5	0.50	0.2221180	0.1487206	0.0648721	0.0731157	0.0735278	0.0822249
6	0.60	0.3137516	0.2221180	0.0822118	0.0913224	0.0917779	0.1013896
7	0.70	0.4255396	0.3137516	0.1013752	0.1114439	0.1119474	0.1225699
8	0.80	0.5596014	0.4255396	0.1225540	0.1336817	0.1342380	0.1459778
9	0.90	0.7182797	0.5596014	0.1459601	0.1582582	0.1588730	0.1718474
10	1.00		0.7182797				

* 비교

	A	B	C	D	E	F	G
1							
2					(e^x)-x-1		
3	i	x_i	2차에서 구한 y_i	4차에서 구한 y_i	실제값	2nd 오차	4th 오차
4	0	0.00	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
5	1	0.10	0.0050000	0.0051708	0.0051709	0.0001709	0.0000001
6	2	0.20	0.0210250	0.0214026	0.0214028	0.0003778	0.0000002
7	3	0.30	0.0492326	0.0498585	0.0498588	0.0006262	0.0000003
8	4	0.40	0.0909021	0.0918242	0.0918247	0.0009226	0.0000005
9	5	0.50	0.1474468	0.1487206	0.1487213	0.0012745	0.0000006
10	6	0.60	0.2204287	0.2221180	0.2221188	0.0016901	0.0000008
11	7	0.70	0.3115737	0.3137516	0.3137527	0.0021790	0.0000011
12	8	0.80	0.4227889	0.4255396	0.4255409	0.0027520	0.0000014
13	9	0.90	0.5561818	0.5596014	0.5596031	0.0034214	0.0000017
14	10	1.00	0.7140808	0.7182797	0.7182818	0.0042010	0.0000021

-> 차수가 높아질수록 오차가 줄어듬!!

Summary

1. Runge-Kutta법 :

   - 도함수의 평균값으로 수치해석
   - 차수가 높을수록 정밀

2. 비쥬얼베이직에디터 호출방법 : [Alt+F11] key

3. 모듈 작성하는 방법 : [메뉴>삽입>모듈]