GOVERNING EQUATION AND BOUNDARY CONDITIONS OF HEAT TRANSFER

introduction

1. Deriving Governing equation

2. Boundary conditions

3. Deriving The bioheat transfer equation

4. Governing Equations for heat condition in Various coordinate systems

5. Problem formulation

Governing Equation for Heat Transfer Derived from Energy Conservation and Fourier's law

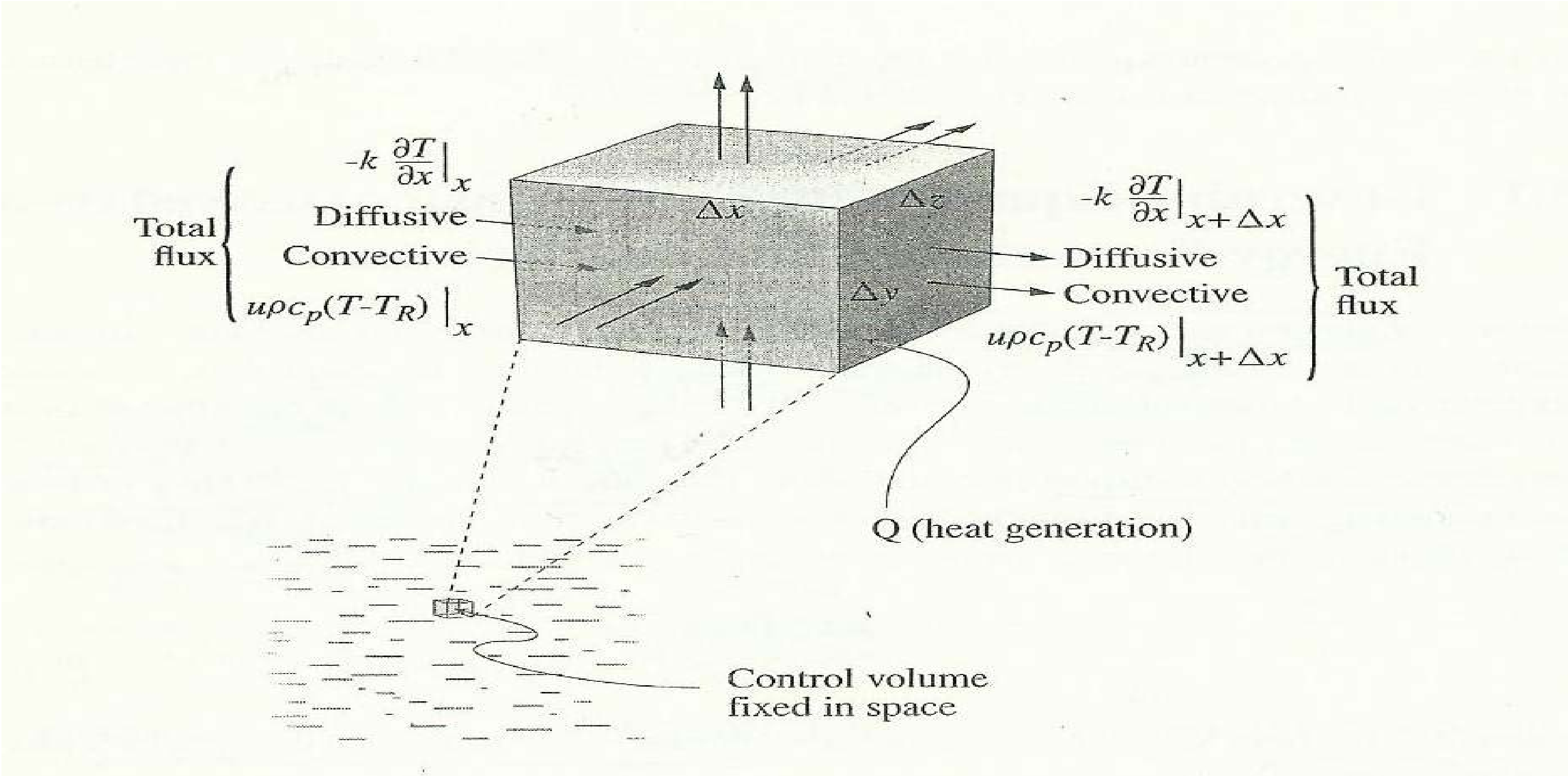

Figure 1. Control volume showing energy inflow and outflow by conduction (diffusion) and convection.

Governing Equation for Heat Transfer Derived from Energy Conservation and Fourier's law

$$\begin{aligned} E &= \dot{m} c_p (T - T_R) \\ &= u A \rho c_p (T - T_R) \end{aligned} \quad (1)$$

$$\frac{E}{A} = u \rho c_p (T - T_R) \quad (2)$$

$$\begin{aligned} &Energy\ In - Energy\ Out + Energy\ Generation \\ &= Energy\ Storage \end{aligned} \quad (3)$$

Governing Equation for Heat Transfer Derived from Energy Conservation and Fourier's law

$$Energy\ In\ during\ time \triangle t = (\underbrace{q''_x \triangle y \triangle z +}_{convection} \underbrace{[u \triangle y \triangle z \rho c_p (T - T_R)]_x}_{conduction}) \triangle t$$

$$Energy\ out\ during\ time \triangle t = (\underbrace{q''_{x+\triangle x} \triangle y \triangle z +}_{conduction} \underbrace{[u \triangle y \triangle z \rho c_p (T - T_R)]_{x+\triangle x}}_{convection}) \triangle t$$

$$Energy\ generated\ during\ time \triangle t = Q \triangle x \triangle y \triangle z \triangle t$$

$$Energy\ stored\ during\ time \triangle t = \triangle x \triangle y \triangle z \rho c_p \triangle T$$

Governing Equation for Heat Transfer Derived from Energy Conservation and Fourier's law

$$\begin{aligned}\triangle t(q''_x \triangle y \triangle z - q''_{x+\triangle x} \triangle y \triangle z \\ + \rho c_p \triangle y \triangle z [u(T - T_R)_x - u(T - T_R)_{x+\triangle x}] \\ + Q \triangle x \triangle y \triangle z) = \rho c_p \triangle x \triangle y \triangle z \triangle T\end{aligned} \quad (4)$$

$$\begin{aligned}-\frac{q''_{x+\triangle x} - q''_x}{\triangle x} - \rho c_p \frac{(u(T - T_R)_{x+\triangle x} - u(T - T_R)_x)}{\triangle x} + Q \\ = \rho c_p \frac{\triangle T}{\triangle t}\end{aligned} \quad (5)$$

Governing Equation for Heat Transfer Derived from Energy Conservation and Fourier's law

$$-\frac{\partial q''_x}{\partial x} - \rho c_p \frac{\partial}{\partial x}(uT) + Q = \rho c_p \frac{\partial T}{\partial t} \tag{6}$$

$$-\frac{\partial}{\partial x}\left(-k\frac{\partial T}{\partial x}\right) - \rho c_p \frac{\partial}{\partial x}(uT) + Q = \rho c_p \frac{\partial T}{\partial t} \tag{7}$$

$$\underbrace{\frac{\partial T}{\partial t}}_{storage} + \underbrace{\frac{\partial (uT)}{\partial x}}_{\substack{flow\,\text{or} \\ convection}} = \underbrace{\frac{k}{\rho c_p}\frac{\partial^2 T}{\partial x^2}}_{conduction} + \underbrace{\frac{Q}{\rho c_p}}_{generation} \tag{8}$$

$$\underbrace{\frac{\partial T}{\partial t}}_{storage} + \underbrace{u\frac{\partial T}{\partial x}}_{\substack{flow\,\text{or} \\ convection}} = \underbrace{\frac{k}{\rho c_p}\frac{\partial^2 T}{\partial x^2}}_{conduction} + \underbrace{\frac{Q}{\rho c_p}}_{generation} \tag{9}$$

Meaning of Each Term in the Governing Equation

$$\underbrace{\rho c_p \frac{\partial T}{\partial t}}_{storage} + \underbrace{\rho c_p \frac{\partial (uT)}{\partial x}}_{convection} = \underbrace{k \frac{\partial^2 T}{\partial x^2}}_{conduction} + \underbrace{Q}_{generation} \qquad (3.10)$$

Term	What does it represent	When can you ignore it
Storage	Rate of change of stored energy	steady state (no variation of temperature with time)
Convection	Rate of net energy transport due to bulk flow	Typically in a solid, with no bulk flow through it
Conduction	Rate of net energy transport due to conduction	Slow thermal conduction in relation to generation or convection. For example, in short periods of microwave heating
Generation	Rate of generation of energy	No internal heat generation due to biochemical reactions, etc.

Examples of Thermal Source (Generation) Term in Biological Systems

A working muscle such as in the heart or limbs produce heat
Fermentation, composting and other biochemical reactions generate heat

Utility of the Energy Equation

- It is very general

1. it is useful for any material.
2. it is useful for any size of shape . similar equations can be derived for other coordinate systems.
3. it is easier to derive the more general equation and simplify.
4. it is safer – as you drop terms you are aware of the reasons.

- Can we make it general?

1. To use with compressible fluids.
2. To use when all properties vary with temperature. We need numerical solutions to solve such problems.
3. To include mass transfer.

Example Solution to Specific Situations: Need for Boundary conditions

$$\underbrace{\left(\frac{\partial T}{\partial t}\right)}_{steady\,state \to 0} = \underbrace{\frac{k}{\rho c_p}\frac{\partial^2 T}{\partial x^2}}_{conduction} + \underbrace{\left(\frac{Q}{\rho c_p}\right)}_{no\ generation \to 0} \qquad (11)$$

$$\frac{d^2 T}{dx^2} = 0 \qquad (12)$$

$$T = C_1 x + C_2 \qquad (13)$$

$$T = T_1 \ at \ x = 0 \qquad (14)$$

$$T = T_2 \ at \ x = L \qquad (15)$$

$$T_1 = C_2 \qquad T_2 = C_1 L + C_2$$

$$T = \frac{T_2 - T_1}{L} x + T_1 \qquad (16)$$

General Boundary conditions

1. Surface temperature is specified

$$T\big|_{x=0} = T_s$$

$$T\big|_{x=0} = 100$$

General Boundary conditions

1. Surface temperature is specified

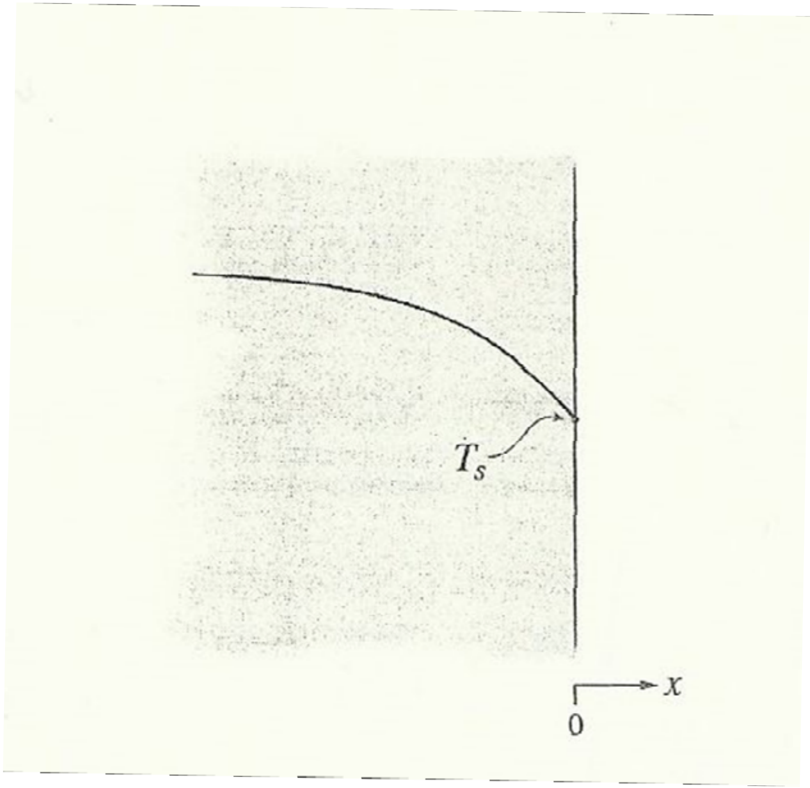

Figure 2. A surface temperature specified boundary condition.

General Boundary conditions

2. Surface heat flux is specified

$$-k\frac{dT}{dx}\Big|_{x=0} = q''_s$$

$$-k\frac{dT}{dx}\Big|_{x=0} = 4000$$

General Boundary conditions

2. Surface heat flux is specified

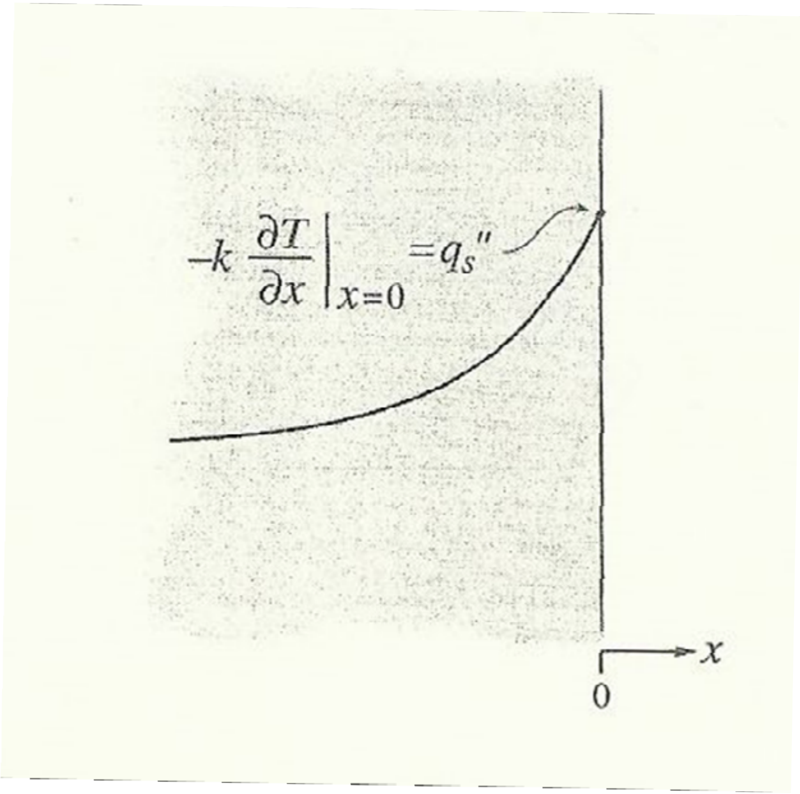

Figure 3. A heat flux specified boundary condition.

General Boundary conditions

2a) special case: Insulated condition

$$-k\frac{dT}{dx}\Big|_{x=0} = 0$$

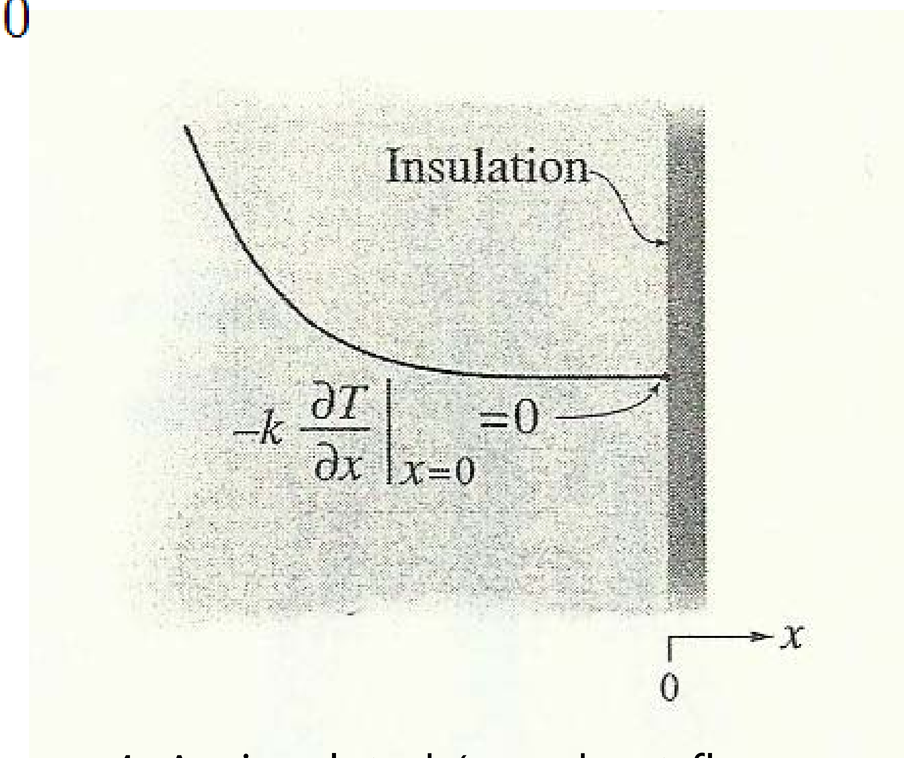

Figure 4. An insulated (zero heat flux specified) boundary condition.

3.2 General Boundary conditions

2b) special case: Symmetry condition

$$-k\frac{dT}{dx}\Big|_{x=L} = 0$$

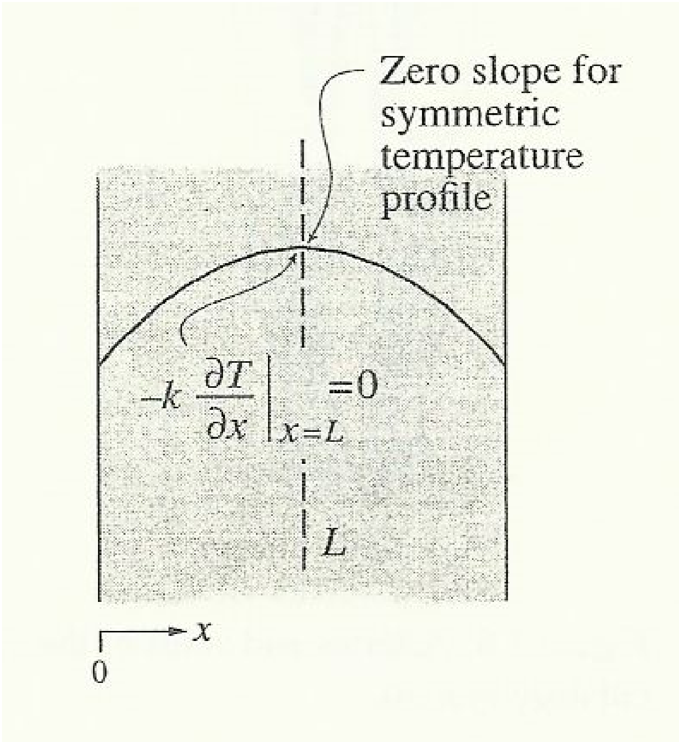

Figure 5. A symmetry (zero heat flux specified) boundary condition at the centerline.

General Boundary conditions

3. Convection at the surface

$$\underbrace{-k\frac{dT}{dx}\Big|_{x=0}}_{heat\ conduction} = \underbrace{h\left(T|_{x=0} - T_{\infty}\right)}_{heat\ convection}$$

$$-k\frac{dT}{dx}\Big|_{x=0} = 50\left(T|_{x=0} - 10\right)$$

General Boundary conditions

3. Convection at the surface

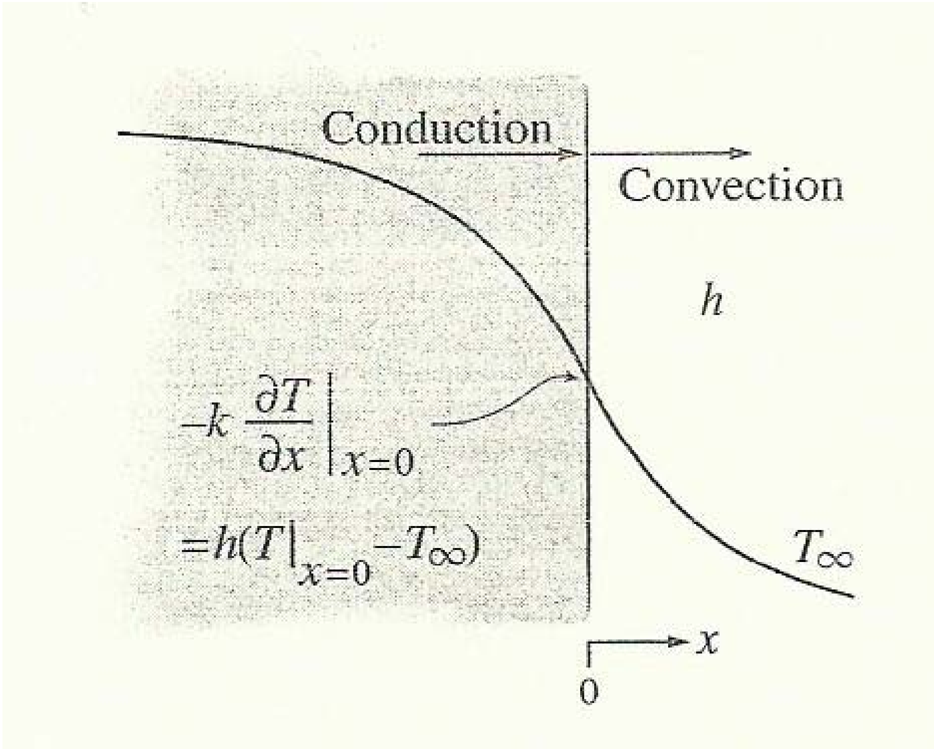

Figure 6. A convection boundary condition.

Summary

-Governing Equation

1. It is a mathematical statement of energy conservation. It is obtained by combining conservation of energy with Fourier 's law for heat conduction.
2. Depending on the appropriate geometry of the physical problem , choose a governing equation in a particular coordinate system from the equations
3. Different terms in the governing equation can be identified with conduction convection , generation and storage. Depending on the physical situation some terms may be dropped.

-Boundary conditions

1. Boundary conditions are the conditions at the surfaces of a body.
2. Initial conditions are the conditions at time $t = 0$.
3. Boundary and initial conditions are needed to solve the governing equation for a specific physical situation.
4. One of the following three types of heat transfer boundary conditions typically exists on a surface:
   (a) Temperature at the surface is specified
   (b) Heat flux at the surface is specified
   (c) Convective heat transfer condition at the surface