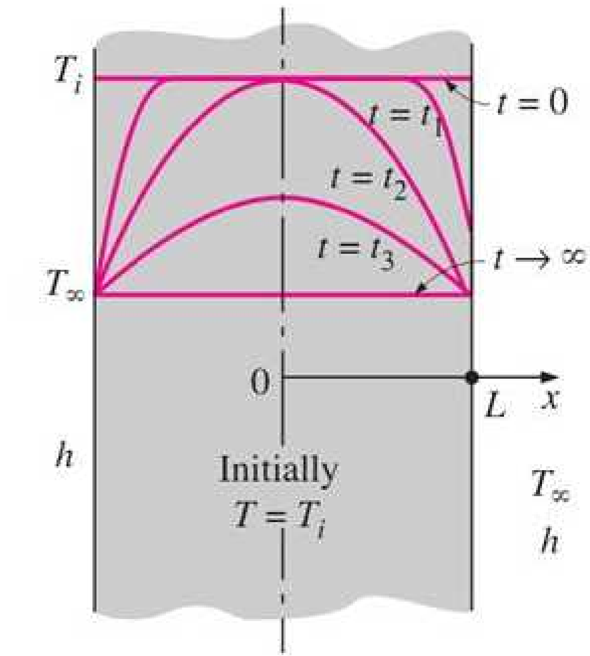
5. Unsteady Heat Conduction (slab)

Assume no heat generation (no heat source)



$x = 0$ $x = s$

i. Heat conduction equation

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2}{\partial x^2} + \frac{\dot{q}}{\rho C_p}$$

I.C. $T(x, 0) = T_a$

B.C. 1) $T(\pm s, t) = T_s$

B.C. 2) $\frac{\partial T(0,t)}{\partial x} = 0$

ii. Solution → Temp. profile

$$\frac{T_s - T}{T_s - T_a} = 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{\left(n + \frac{1}{2}\right)\pi} e^{-\left(n+\frac{1}{2}\right)\pi\left(\frac{\alpha t}{s^2}\right)} \cos\left\{\left(n + \frac{1}{2}\right)\pi\left(\frac{x}{s}\right)\right\}$$

iii. Mean Temp. $\bar{T}(t)$

$$\frac{T_s - \bar{T}(t)}{T_s - T_a} = \frac{8}{\pi^2}\left\{e^{-\left(\frac{\pi}{2}\right)^2\left(\frac{\alpha t}{s^2}\right)} + \frac{1}{9}e^{-9\left(\frac{\pi}{2}\right)^2\left(\frac{\alpha t}{s^2}\right)} + \frac{1}{25}e^{-25\left(\frac{\alpha t}{s^2}\right)} + \cdots\right\}$$

Fourier's number, Fo, (N_{Fo})

$$Fo = \frac{\alpha t}{s^2}$$

if $Fo\left(=\frac{\alpha t}{s^2}\right), > 0.1$ then,

$$\frac{T_s - \bar{T}(t)}{T_s - T_a} = \frac{8}{\pi^2}e^{-\left(\frac{\pi}{2}\right)^2\left(\frac{\alpha t}{s^2}\right)} = \frac{8}{\pi^2}e^{-\left(\frac{\pi}{2}\right)^2 Fo}$$

$$T\Big|_{T_a \to \bar{T}} \frac{1}{\alpha}\left(\frac{2s}{\pi}\right)^2 \ln\left\{\frac{8}{\pi^2}\left(\frac{T_s - T_a}{T_s - \bar{T}}\right)\right\}$$

iv. Total heat transferred Q_T

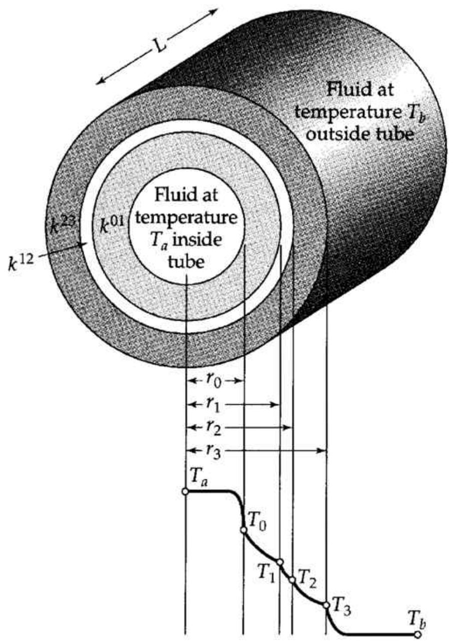
$$Q_T = m\,C_p(\bar{T} - T_a) = \rho\{A\,s\}C_p(\bar{T} - T_a)$$

Or total heat transferred per unit area

$$\begin{aligned} Q_T/A &= \frac{\rho\{A\,s\}C_p(\bar{T} - T_a)}{A} \\ &= \rho s\,C_p(\bar{T} - T_a) \end{aligned}$$

5. Unsteady Heat Conduction (cylinder)

Assume no heat generation (no heat source)



i. Heat conduction equation

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial}{r\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\dot{q}}{\rho C_p}$$

I.C. $T(r,0) = T_a$

B.C. 1) $T(R,t) = T_s$

B.C. 2) $\frac{\partial T(0,t)}{\partial r} = 0$

ii. Solution → Temp. profile

$$\frac{T_s - T(r,t)}{T_s - T_a} = \sum_{n=1}^{\infty} 2\, e^{-m_i\left(\frac{\alpha t}{R^2}\right)} \frac{J_o\left[m_i \frac{r}{R}\right]}{m_i\, J_i(-m_i)}$$

m_i: eigenvalues of Bessel function $J_o(m_i) = 0$

iii. Mean Temp. $\bar{T}(t)$

Fourier's number, Fo, (N_{Fo})

$$Fo = \frac{\alpha t}{s^2}$$

$$\frac{T_s - \bar{T}(t)}{T_s - T_a} = \sum_{n=1}^{\infty} \frac{4}{m_i^4} e^{-m_i^2\left(\frac{\alpha t}{R^2}\right)}$$

if $Fo\left(= \frac{\alpha t}{s^2}\right), > 0.1$ then,

$$\frac{T_s - \bar{T}(t)}{T_s - T_a} \cong 0.692 e^{-5.78\left(\frac{\alpha t}{s^2}\right)} = 0.692 e^{-5.78\ Fo}$$

or

$$t = \frac{R^2}{5.78\ \alpha} \ln \frac{T_s - T_a}{T_s - \bar{T}}$$

iv. Total heat transferred Q_T

$$Q_T = m\ C_p(\bar{T} - T_a) = \rho(\pi R^2 L) C_p(\bar{T} - T_a)$$

Also, total heat transferred per unit area

$$Q_T/A = \frac{\rho(\pi R^2 L) C_p(\bar{T} - T_a)}{2\pi R L}$$

Cf. for slab

$$Q_T/A = \rho s\, C_p(\bar{T} - T_a)$$

Ex. 10.10) **A layer of pulverized cork (insulator)**

$$\rho = 900\ kg/m^3 \qquad T_s = 121.1℃$$
$$k = 0.13\ W/m℃ \qquad T_a = 21.1℃$$
$$C_p = 1.67\ J/g℃ = 1.67\times10^3\ J/kg℃$$

$$s = \frac{2.54}{2}cm = 1.27cm = 1.27\times10^{-2}m$$

$$\alpha\left(\equiv\frac{k}{\rho C_p}\right) = \frac{0.13\left(\frac{W}{m℃}\right)}{900\left(\frac{kg}{m^3}\right)1.67\times10^3\left(\frac{J}{kg\ ℃}\right)} = 8.65\times10^{-8}\ (m^2/s)$$

$\bar{T} = 98.9℃$ 라면 그 때의 t ?

$$\frac{T_s-\bar{T}}{T_s-T_a} = \frac{121.1-98.9}{121.1-21.1} = 0.222$$

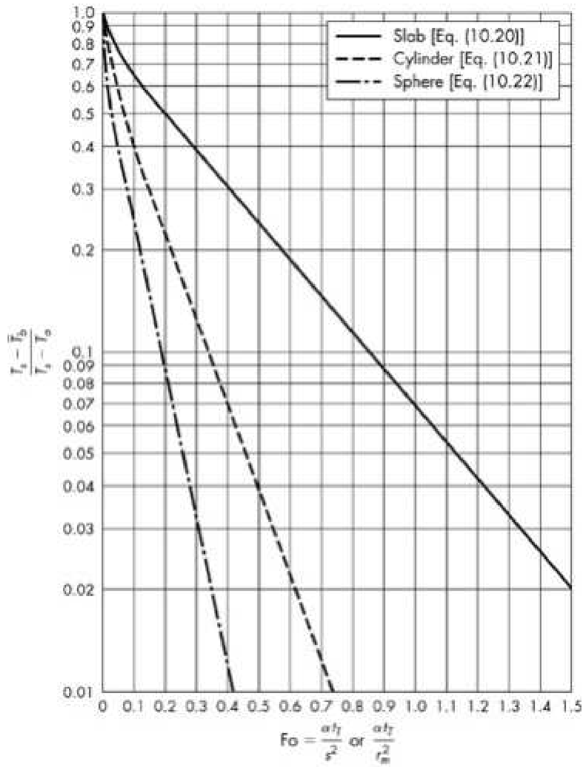


Fig. 10.5 에서 y 축의 값이 0.222 면 $Fo \approx 0.52$

$$Fo = 0.52 = \frac{\alpha t}{s^2}$$

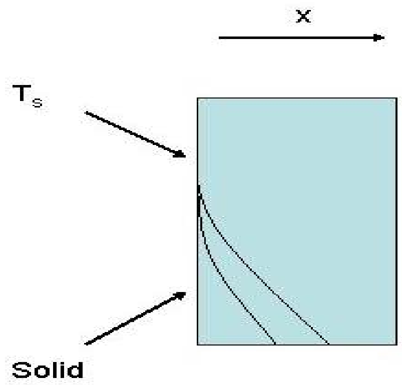
$$t = 0.52\frac{s^2}{\alpha} = 0.52\frac{(1.27\times10^{-2})^2\ m^2}{8.65\times10^{-8}\ (m^2/s)} = 969.6\ s$$

$$Q_T = \rho\{A\ s\}C_p(\bar{T} - T_a)$$

$$\frac{Q_T}{A} = \rho s\, C_p(\bar{T} - T_a)$$
$$= 900\left(\frac{kg}{m^3}\right)1.67 \times 10^3\ \left(\frac{J}{kg\ ℃}\right)1.27 \times 10^{-2}\ (m)(98.9 - 21.1)℃$$
$$= 1686.43\ kJ\,/m^2$$

5. Unsteady Heat Conduction

2) Semi-infinite solids (constant surface temperature)



1. Heat Conduction Equation

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2}{\partial x^2}$$

I.C. $T(x, 0) = T_a$
B.C. 1) $T(0, t) = T_s$
B.C. 2) $T(\infty, t) = T_a$

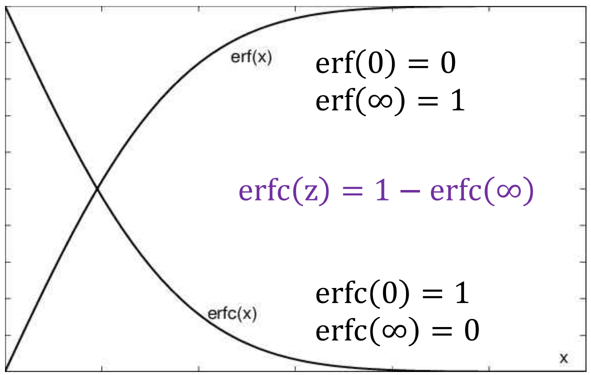
2. Solution → Temperature distribution (profile)

$$\frac{T_s - T}{T_s - T_a} = \frac{2}{\sqrt{\pi}} \int_0^z e^{-z^2} dz$$

where, $z = \frac{x}{2\sqrt{\alpha\, x}}$

Error f unction (Gauss Error Integration)

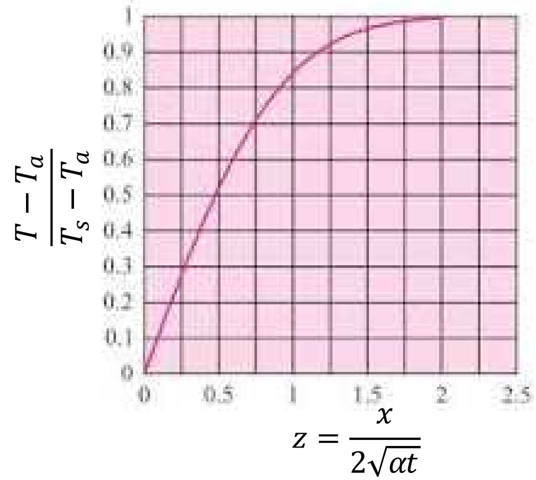
$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-z^2} dz$$



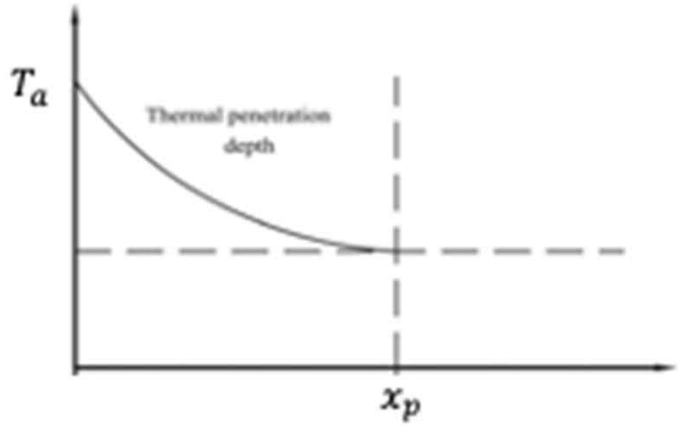
5. Unsteady Heat Conduction

3) Penetration depth, x_p

Distance from the surface at which the temperature change is 1% of the initial change in the surface change



$$\frac{T - T_a}{T_s - T_a} = 0.01 \qquad \frac{T_s - T}{T_s - T_a} = 0.99$$



$\mathrm{erf}(z) = 0.99$ at $z = 1.82 \rightarrow (z_p)$

i.e. $z_p = \frac{x_p}{2\sqrt{\alpha t}} = 1.82$

$$x_p = 3.64\sqrt{\alpha t}$$

5. Unsteady Heat Conduction

4) Total heat transferred, Q_T

$$Q_T = \int_0^t q\Big|_{x=0} dt = \int_0^t \left[-kA\frac{dT}{dx}\right]_{x=0} dt$$

$$\Rightarrow \frac{Q_T}{A} \int_0^t \left[-kA\frac{dT}{dx}\right]_{x=0} dt$$

$$\frac{dx}{dx}\frac{dT}{dz} = -\frac{2}{\sqrt{\pi}}(T_s - T_a)e^{-z^2}$$

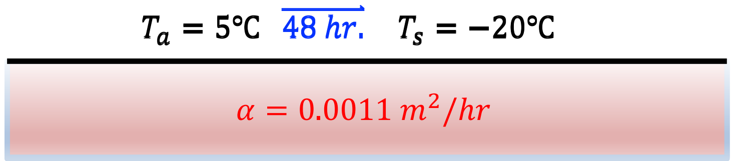
$$\frac{dx}{dz}\frac{dT}{dx} = -\frac{2}{\sqrt{\pi}}(T_s - T_a)e^{-z^2}$$

$$\frac{dT}{dx}\Big|_{x=0} = \left(\frac{dz}{dx}\right)\left[-\frac{2}{\sqrt{\pi}}(T_s - T_a)e^{-z^2}\right]\Bigg|_{z=0} = -\frac{T_s - T_a}{\sqrt{\pi\,\alpha\,t}}$$

$$\frac{Q_T}{A} = k\frac{T_s - T_a}{\sqrt{\pi\,\alpha}}\int_0^t \frac{1}{\sqrt{t}}dt$$

$$= 2k(T_s - T_a)\sqrt{\frac{t}{\pi\alpha}} = 2k(T_s - T_a)\sqrt{\frac{\rho C_p k}{\pi}t}$$

Ex. 10.7)



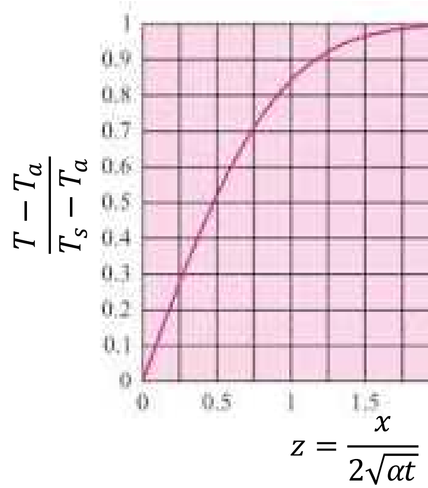


fig. 10.10

a) $x|_{T=0°C} = ?$

$$\frac{T_s - T}{T_s - T_a}(= \mathrm{erf}(z)) = \frac{(-20) - 0}{(-20) - 5} = 0.8 \quad \Rightarrow \quad z\left(= \frac{x}{2\sqrt{\alpha t}}\right) = 0.91$$

From fig. 10.10

$$\Rightarrow \quad x = 0.91 \times 2\sqrt{\alpha t} = 0.91 \times 2\sqrt{0.0011(m^2/hr) \times 48(hr)} = 0.418\ m$$

a) $x_p = ?$ $\qquad \frac{T_s - T}{T_s - T_a} = 0.99 \ \Rightarrow T = 4.75°C \qquad x_p\big|_{T=4.75°C}$

$$x_p = 3.64\sqrt{\alpha t} = 3.64\sqrt{0.0011(m^2/hr) \times 48(hr)} = 0.836\ m$$

Problems good to solve : (Probs.) 10.1, 10.2, 10.4, 10.5, 10.6, 10.7 and 10.10