3.3 NECESSARY CONDITION OF OPTIMALITY FOR CONSTRAINED OPTIMIZATION PROBLEMS

Constrained Optimization Problems

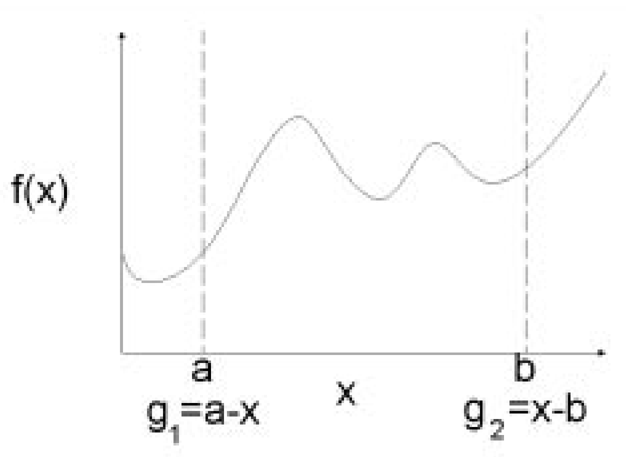
Consider

$$\min_{x \in \mathbf{R}} f(x)$$

subject to

$$g_1(x) = a - x \le 0$$

$$g_2(x) = x - b \le 0$$



$$\Downarrow$$

$\nabla f(x^*) = 0$ is not the necessary condition of optimality anymore.

Lagrange Multiplier

Consider

$$\min_{x \in \mathbf{R}^n} f(x)$$

subject to

$$h(x) = 0$$

At the minimum, the m constraint equations must be saisfied

$$h(x^*) = 0$$

Moreover, at the minimum,

$$df(x^*) = \frac{df}{dx}(x^*)dx = 0$$

must hold in any feasible direction.

Feasible direction, $dx^\dagger$, must satisfy

$$dh(x^*) = \frac{dh}{dx}(x^*)dx^\dagger = 0$$

$$\Updownarrow$$

For any $y = \Sigma_{i=1}^m a_i \frac{dh_i}{dx}(x^*)$,

$$y^T dx^\dagger = 0$$

Lagrange Multiplier (Continued)

$df(x^*) = \frac{df}{dx}(x^*)dx^\dagger = 0$ must hold

$$\Downarrow$$

$\frac{df}{dx}(x^*)$ is linearly dependent on $\{\frac{dh_i}{dx}(x^*)\}_{i=1}^m$

$$\Downarrow$$

$\exists\ \{\lambda_i\}_{i=1}^m$ such that

$$\frac{df}{dx}(x^*) + \sum_{i=1}^m \lambda_i \frac{dh_i}{dx}(x^*) = 0$$

$$\Downarrow$$

Necessary Condition of Optimality:

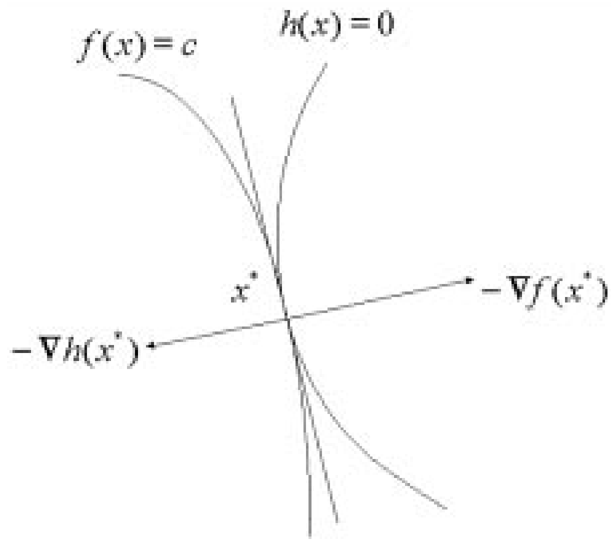
$$h(x^*) = 0 \quad m \text{ equations}$$

$$\frac{df}{dx}(x^*) + \sum_{i=1}^m \lambda_i \frac{dh_i}{dx}(x^*) = 0 \quad n \text{ equations}$$

where λ_i's are called Lagrange Multipliers.

($n + m$ equations and $n + m$ unknowns)

Lagrange Multiplier (Continued)



x* is a local minimum

Lagrange Multiplier (Continued)

Example: Consider

$$\min_{x \in \mathbf{R}^n} \frac{1}{2} x^T H x + g^T x$$

subject to

$$Ax - b = 0$$

The necessary condition of optimality for this problem is

$$[\nabla f(x^*)]^T + [\nabla h(x^*)]^T \lambda = Hx^* + g + A^T \lambda = 0$$

$$h(x^*) = Ax^* - b = 0$$

$$\Downarrow$$

$$Hx^* + A^T \lambda = -g$$

$$Ax^* = b$$

$$\Downarrow$$

$$\begin{bmatrix} H & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} x^* \\ \lambda \end{bmatrix} = \begin{bmatrix} -g \\ b \end{bmatrix}$$

If $\begin{bmatrix} H & A^T \\ A & 0 \end{bmatrix}$ is invertible,

$$\begin{bmatrix} x^* \\ \lambda \end{bmatrix} = \begin{bmatrix} H & A^T \\ A & 0 \end{bmatrix}^{-1} \begin{bmatrix} -g \\ b \end{bmatrix}$$

Kuhn-Tucker Condition

Let x^* be a local minimum of

$$\min f(x)$$

subject to

$$h(x) = 0$$

$$g(x) \leq 0$$

and suppose x^* is a regular point for the constraints. Then $\exists\ \lambda$ and μ such that

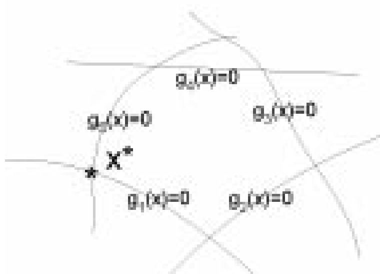
$$\nabla f(x^*) + \lambda^T \nabla h(x^*) + \mu^T \nabla g(x^*) = 0$$

$$\mu^T g(x^*) = 0$$

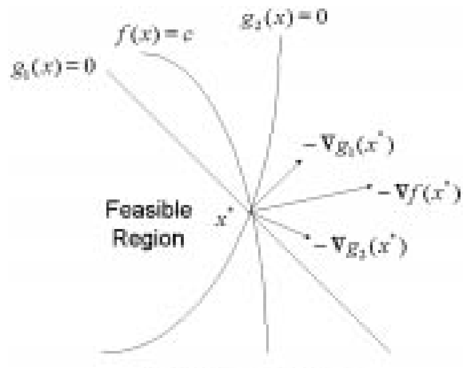
$$h(x^*) = 0$$

$$\mu \geq 0$$

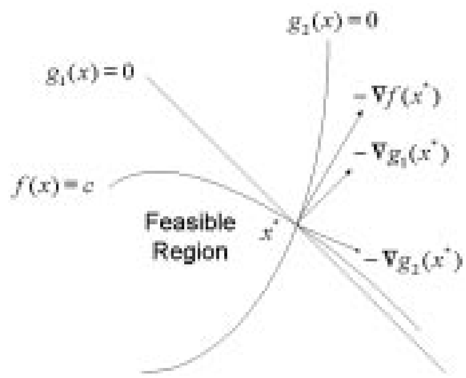
$g_i(x^*) < 0 \Rightarrow \mu_i = 0$



Kuhn-Tucker Condition(Continued)



x* is a local minimum



x* is not a local minimum

Kuhn-Tucker Condition (Continued)

Example: Consider

$$\min_{x \in \mathbf{R}^n} \frac{1}{2} x^T H x + g^T x$$

subject to

$$Ax - b = 0$$

$$Cx - d \leq 0$$

The necessary condition of optimality for this problem is

$$[\nabla f(x^*)]^T + [\nabla h(x^*)]^T \lambda + [\nabla g(x^*)]^T \mu = Hx^* + g + A^T \lambda + C^T \mu = 0$$

$$g(x^*)^T \mu = (x^{*T} C^T + d^T)\mu = 0$$

$$h(x^*) = Ax^* - b = 0$$

$$\mu \geq 0$$

$$\Downarrow$$

$$Hx^* + A^T \lambda + C^T \mu = -g$$

$$x^{*T} C^T \mu + d^T \mu = 0$$

$$Ax^* = b$$

$$\mu \geq 0$$