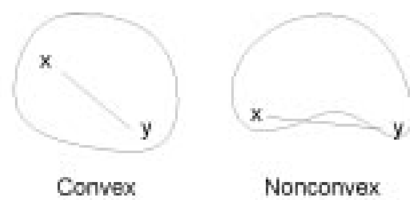
3.4 CONVEX OPTIMIZATION

Convexity

Convex set: $C \subset \mathbf{R}^n$ is convex if

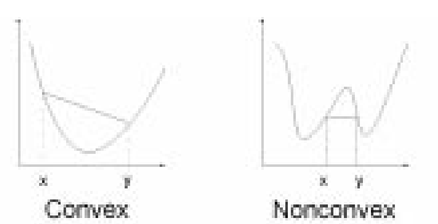
$$x, y \in C, \quad \lambda \in [0,1] \quad \Rightarrow \quad \lambda x + (1-\lambda) y \in C$$



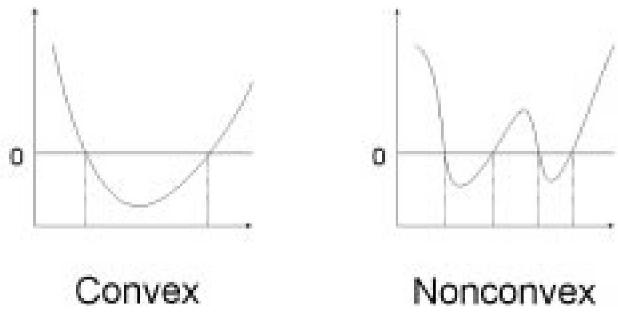
Convex Functions: $f : \mathbf{R}^n \rightarrow \mathbf{R}$ is convex if

$$x, y \in \mathbf{R}^n, \quad \lambda \in [0,1]$$

$$f(\lambda x + (1-\lambda) y) \leq \lambda f(x) + (1-\lambda) f(y)$$



Convexity (Continued)



Notice that $\{x : g(x) \leq 0\}$ is convex if g is convex.

Theorem: If f and g are convex any local optimum is globally optimal.

Linear Programs

$$\min_{x \in \mathbf{R}^n} a^T x$$

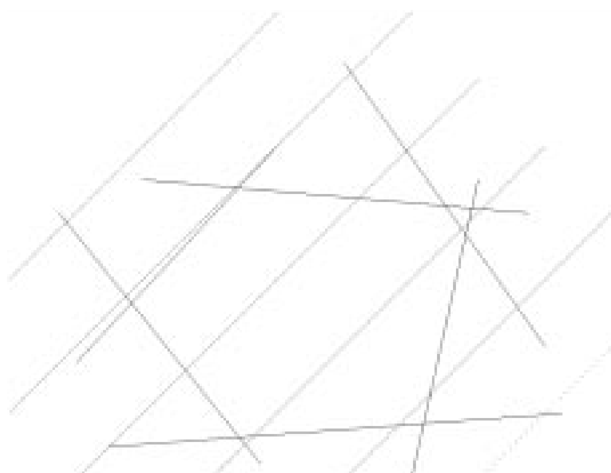
subject to

$$Bx \leq b$$

Linear program is a convex program.

Feasible basic solution: feasible solution that satisfies n of the constraints as equalities.

Fact: If an optimal solution exists, there exists a feasible basic solution that is optimal.

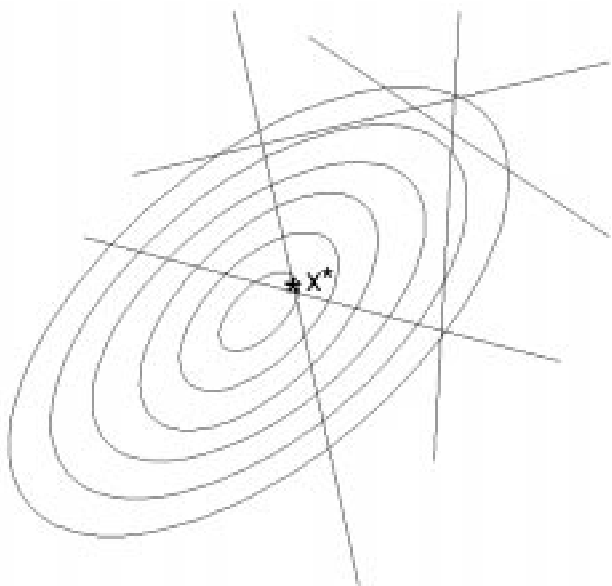


Quadratic Programs

$$\min_{x \in \mathbf{R}^n} \frac{1}{2} x^T H x + g^T x$$

subject to

$$Ax \leq b$$



Quadratic program is convex if H is positive semi-definite.

3.5 ALGORITMS FOR CONSTRAINED OPTIMIZATION PROBLEMS

Algorithms for Linear Program

Simplex Method

Motivation: There always exists a basic optimal solution.

Main Idea:

- Find a basic solution.
- Find another basic solution with lower cost function value.
- Continue until another basic solution with lower cost function value cannot be found.

Simplex algorithm always finds a basic optimal solution.

Algorithms for Linear Program (Continued)

Interior Point Method

Main Idea:

- Define barrier function:

$$B = -\sum_{i=1}^{m} \frac{1}{c_i^T x - b_i}$$

- Form the unconstrained problem:

$$\min_x a^T x + \frac{1}{K} B(x)$$

- Solve the unconstrained problem using Newton method.
- Increase K and solve the unconstrained problem again until the solution converges.

- Remarkably, problems seem to converge between 5 to 50 Newton steps regardless of the problem size.
- Can exploit structures of the problem (e.g. sparsity) to reduce computation time per Newton step.
- Can be extended to general nonlinear convex problems such as quadratic programs.

Algorithms for Quadratic Program

Active Set Method

Main Idea:

- Determine the active constraints and set them as equality constraints.
- Solve the resulting problem.
- Check the Kuhn-Tucker condition that is also suffucuent for QP.
- If Kuhn-Tucker condition is not satisfied, try another set of active constraints.

Interior Point Method

- The main idea of interior point method for QP is the same as that for LP.

Generalized Reduced Gradient Method for Constrained Nonlinear Programs

Main idea:

1. Linearize the equality constraints that are possibly obtained adding slack variables
2. Solve the resulting linear equations for m variables
3. Apply the steepest descent method with respect to $n-m$ variables

Linearization of Constraints:

$$\nabla_y h(y,z)dy + \lambda^T \nabla_z h(y,z)dz = 0$$

$$\Downarrow$$

$$dy = -[\nabla_y h(y,z)]^{-1} \lambda^T \nabla_z h(y,z)dz$$

Generalized Reduced Gradient of Objective Function:

$$df(y,z) = \nabla_y f(y,z)dy + \lambda^T \nabla_z f(y,z)dz$$

$$= [\lambda^T \nabla_z f(y,z) - \nabla_y f(y,z)[\nabla_y h(y,z)]^{-1} \lambda^T \nabla_z h(y,z)]dz$$

$$\Downarrow$$

$$r = \frac{df}{dz} = \lambda^T \nabla_z f(y,z) - \nabla_y f(y,z)[\nabla_y h(y,z)]^{-1} \lambda^T \nabla_z h(y,z)$$

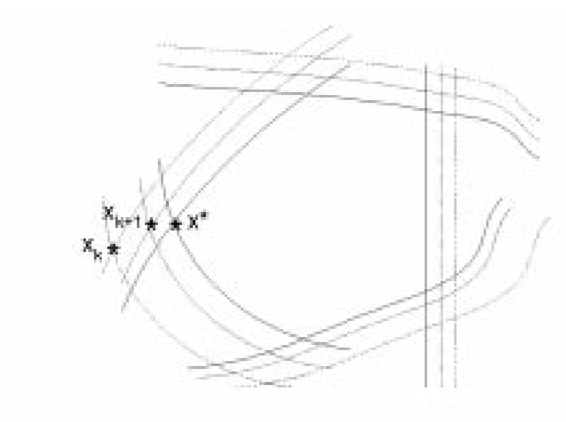
Penalty Method for Constrained Nonlinear Programs

Main idea: Instead of forcing the constraints, penalize the violation of the constraints in the objective.

$$\min_x f(x) - c_k g(x) \qquad (P_k)$$

where $c_k > 0$.

Theorem: Let x_k be the optimal solution of (P_k). Then as $c_k \to \infty$, $x_k \to x^*$.



Successive QP Method
for Constrained Nonlinear Programs

Main idea:

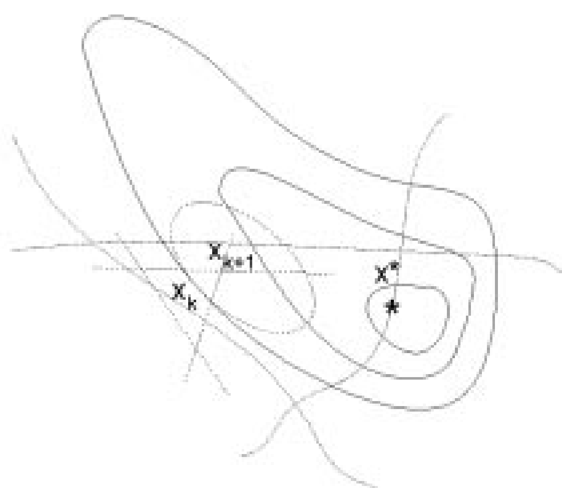
1. Approximate the object function by quadradic function and constraints linear function.
2. Solve the resulting quadratic problem

Approximate Quadratic Program:

$$\min \nabla f dx + \frac{1}{2} dx^T \nabla^2 f dx$$

subject to

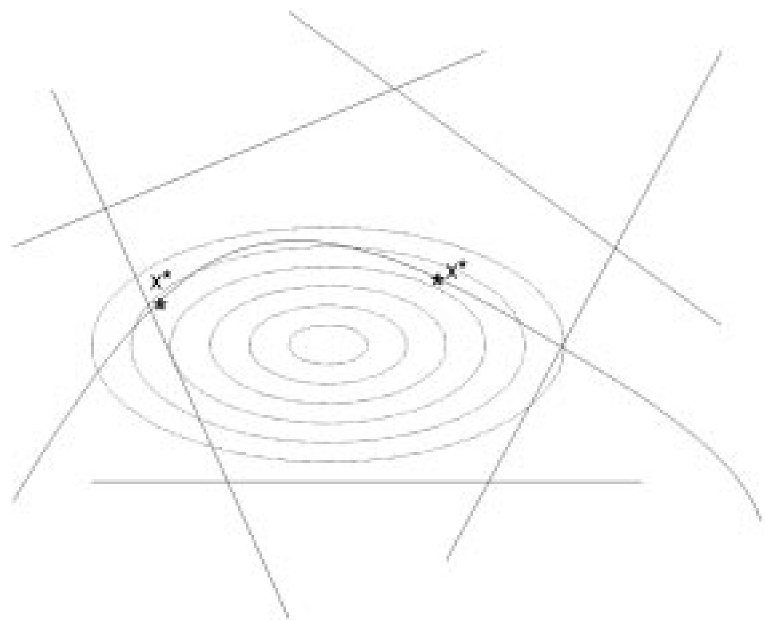
$$g(x) + \nabla g(x) dx \leq 0$$



Nonconvex Programs

The aforementioned optimization algorithms indentify only one local optimum.

However, a nonconvex optimization problem may have a number of local optima.



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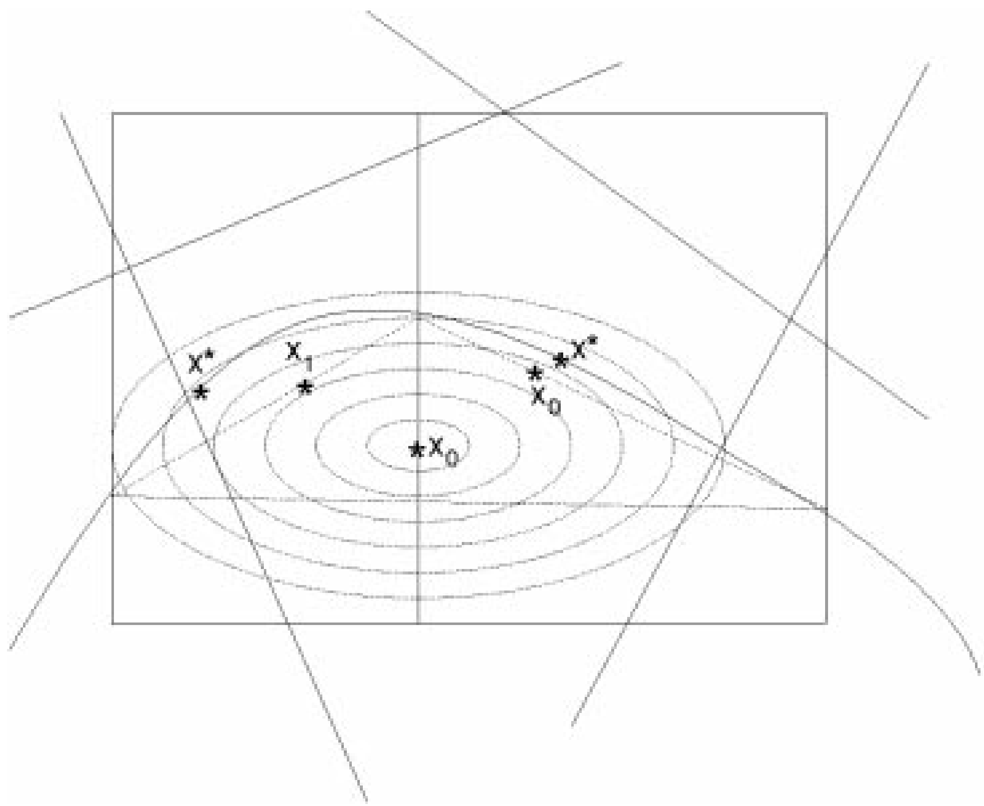
Algorithms that indentifies a global optimum are necessary

A Global Optimization Algorithm for Noconvex Programs

Branch and bound type global optimization algorithm:



- Branching Step: split the box at the optimum
- Bounding Step: find the box where the optimum is lowest